The wave equation in cylindrical co-ordinates for E_z is

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} = -\omega^2 \mu \varepsilon E_z \rightarrow (1)$$

Like rectangular case,

$$E_z = P(\rho)\, Q(\phi)\, e^{-\bar{\gamma} z} = E_z^{\circ} e^{-\bar{\gamma} z} \rightarrow (2)$$

where P is the function of ρ alone
Q is the function of ϕ alone

Subs eq (2) for E_z° in eq (1)

$$\frac{\partial^2 (PQ)}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 (PQ)}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}(PQ) + \frac{1}{\rho}\frac{\partial}{\partial \rho}(PQ) + \omega^2 \mu \varepsilon PQ = 0$$

where $\frac{\partial^2}{\partial z^2} = \bar{\gamma}^2$

$$Q\frac{\partial^2 P}{\partial \rho^2} + \frac{P}{\rho^2}\frac{\partial^2 Q}{\partial \phi^2} + (\bar{\gamma}^2 + \omega^2 \mu \varepsilon) PQ + \frac{Q}{\rho}\frac{\partial P}{\partial \rho} = 0$$

$$Q\frac{\partial^2 P}{\partial \rho^2} + \frac{P}{\rho^2}\frac{\partial^2 Q}{\partial \phi^2} + h^2 PQ + \frac{Q}{\rho}\frac{\partial P}{\partial \rho} = 0 \rightarrow (3)$$

Dividing eq (3) by PQ

$$\frac{1}{P}\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{Q\rho^2}\frac{\partial^2 Q}{\partial \phi^2} + h^2 + \frac{1}{P\rho}\frac{\partial P}{\partial \rho} = 0 \rightarrow (4)$$

Eqn (4) can be broken up into two differential eqns.

$$\frac{1}{Q\rho^2}\frac{\partial^2 Q}{\partial \phi^2} = -\frac{n^2}{\rho^2} \rightarrow (5)$$

$$\frac{1}{P}\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{P\rho}\frac{\partial P}{\partial \rho} + h^2 = \frac{n^2}{\rho^2} \rightarrow (6)$$

from eq (5)

$$\frac{\partial^2 Q}{\partial \phi^2} = \frac{-n^2}{\cancel{\rho^2}} \times Q\cancel{\rho^2}$$

$$\frac{\partial^2 Q}{\partial \phi^2} = -n^2 Q \rightarrow (7)$$

$$\frac{\partial^2 Q}{\partial \phi^2} + n^2 Q = 0$$

$$(m^2 + n^2) Q = 0$$

$$m^2 + n^2 = 0$$

$$m^2 = -n^2$$

$$m = \pm in$$

The solution of eq (7) is

$$Q = A_n \cos n\phi + B_n \sin n\phi \rightarrow (8)$$

Eq (6) is multiplied by P, we get

$$\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial P}{\partial \rho} + h^2 P = \frac{n^2 P}{\rho^2}$$

$$\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial P}{\partial \rho} + \left(h^2 - \frac{n^2}{\rho^2}\right) P = 0 \rightarrow (9)$$

Dividing by h^2 of eq (9)

$$\frac{\partial^2 P}{\partial (\rho h)^2} + \frac{1}{\rho h}\frac{\partial P}{\partial (\rho h)} + \left(1 - \frac{n^2}{(\rho h)^2}\right) P = 0 \rightarrow (1$$

Eqn. is the differential eqn which is the standard form of Besselé eqn. in terms of (ρh).

The Besselé eqn is

$$\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial P}{\partial \rho} + \left(1 - \frac{n^2}{\rho^2}\right) P = 0$$

The soln. of eq (10) is

$$P(\rho h) = J_n(\rho h) \rightarrow (11)$$

$J_n(\rho h)$ is Bessel's function of the first kind of order n.

subs. the soln in eq (2)

$$E_z = J_n(\rho h)\left[A_n \cos n\phi + B_n \sin n\phi\right] e^{-\bar{\gamma} z} \rightarrow (12)$$

III^{ly} $H_z = J_n(\rho h)\left[C_n \cos n\phi + D_n \sin n\phi\right] e^{-\bar{\gamma} z} \rightarrow (13)$

$[B_n \,\&\, D_n = 0]$

TM and TE waves in circular guides

As in the case of rectangular guides, the waves in circular waveguides are also divided into TE and TM waves.

For TM waves

$$\boxed{H_z = 0} \rightarrow (1)$$

The wave equation for E_z is used.

The boundary condition require that E_z must vanish at the surface of the guide.

From E_z equation to satisfy the boundary condition,

$$\boxed{J_n(ha) = 0} \rightarrow (2)$$ where a is the radius of the guide.

There are infinite number of possible TM waves corresponding to the infinite numbers of roots of eqn. (2)

The few roots are

$$(ha)_{01} = 2.405 \quad , \quad (ha_{11}) = 3.85$$
$$(ha)_{02} = 5.52 \quad , \quad (ha)_{12} = 7.02 \quad \rightarrow (3)$$

First subscript → the value of n.
Second subscript → roots in their order of magnitude.
The various TM waves will be referred as

TM_{01}, TM_{02} etc.

$$\text{since } \bar{\gamma} = \sqrt{h^2 - \omega^2\mu\varepsilon} \qquad (\because h^2 = \bar{\gamma}^2 + \omega^2\mu\varepsilon)$$

$$\bar{\beta}_{nm} = \sqrt{\omega^2\mu\varepsilon - h_{nm}^2} \quad \rightarrow (4)$$

The cut off frequency or critical frequency below which transmission of a wave will not occur is

$$f_c = \frac{h_{nm}}{2\pi\sqrt{\mu\varepsilon}} \quad \rightarrow (5)$$

$$\text{where } h_{nm} = \frac{(ha)_{nm}}{a} \quad \rightarrow (6)$$

The phase velocity

$$V = \frac{\omega}{\bar{\beta}} = \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - h_{nm}^2}} \quad \rightarrow (7)$$

The basic equations for TM waves in circular guides are.

$$H_\rho = \frac{j\omega\varepsilon}{h^2\rho}\frac{\partial E_z}{\partial\phi} \qquad E_\rho = \frac{-\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial\rho}$$

$$H_\phi = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial\rho} \qquad E_\phi = \frac{-\bar{\gamma}}{\rho}\frac{\partial E_z}{\partial\phi}$$

Subs $E_z = J_n(\rho h)\, A_n \cos n\phi\, e^{-\bar{\gamma} z}$

$$H_\rho = \frac{j\omega\varepsilon}{h^2\rho}\frac{\partial}{\partial\phi}\left[J_n(\rho h)\, A_n \cos n\phi\, e^{-\bar{\gamma} z}\right]$$

$$= \frac{j\omega\varepsilon}{h^2\rho} J_n(\rho h)\, A_n\, [-\sin n\phi] \times n\, e^{-\bar{\gamma} z}$$

$$H_\rho = \frac{-j\omega\varepsilon n}{h^2\rho} J_n(\rho h)\, A_n \sin n\phi\, e^{-\bar{\gamma} z} \rightarrow (8)$$

IIIly other fields are

$$H_\phi = \frac{-j A_n \omega\varepsilon}{h} J_n'(\rho h) \cos n\phi$$

$$E_\rho = \frac{\bar{\beta}}{\omega\varepsilon} H_\phi$$

$$E_\phi = -\frac{\bar{\beta}}{\omega\varepsilon} H_\rho$$

For TE waves ($D_n = 0$)

$$\therefore H_z = J_n(\rho h)\, C_n \cos n\phi\, e^{-\bar{\gamma} z} \rightarrow (1)$$

The B.C for TM waves is $E_\phi = 0$ at $\rho = a$

E_ϕ is proportional to $\frac{\partial H_z}{\partial \rho}$ & therefore $J_n'(\rho h)$

$\therefore$ The B.C for TM wave is

$$\boxed{J_n'(ha) = 0} \rightarrow (1)$$

The fields are

$$H_z^o = C_n J_n(\rho h) \cos n\phi$$

$$H_\rho^o = \frac{-j\bar{\beta}}{h} C_n J_n'(\rho h) \cos n\phi$$

$$H_\phi^o = j\frac{n\bar{\beta}C_n}{h^2\rho} J_n(\rho h) \sin n\phi$$

$$E_\rho^o = \frac{\omega\mu}{\beta} H_\phi^o$$

$$E_\phi^o = -\frac{\omega\mu}{\bar{\beta}} H_\rho^o$$

The roots of eq (1) are

$$(ha)'_{01} = 3.83 \; , \; (ha)'_{11} = 1.84$$

$$(ha)'_{02} = 7.02 \; , \; (ha)'_{12} = 5.33$$

The corresponding TE waves are reffered as,

TE_{01} , TE_{11} , TE_{02} & TE_{12} etc.

The eqn. for f_c, $\bar{\beta}$, $\bar{\lambda}$ & $\bar{v}$ are identical to those for TM waves.

The dominant modes are

$$\boxed{TM_{01} \; \& \; TE_{11}}$$

(The lowest root $TM_{01} = 2.405$ $TE_{11} = 1.84$)