Recurrence Relation

Let $\{a_n\}$ be a sequence of real numbers with a_n as the n^{th} term.

A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses interms of one or more earlier terms ie, $a_1, a_2, \ldots a_{n-1}$ for all integers n with $n \geq n_0$

Eg: Fibonacci series: $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ which can be represented by the recurrence relation

$f_n = f_{n-1} + f_{n-2}$, $n \geq 2$ with $f_0 = 0$, $f_1 = 1$

Homogeneous Recurrence Relation

A linear homogeneous recurrence relation of degree K with constant coefficients is a recurrence relation of the form, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_K a_{n-K}$, where $c_1, c_2, \ldots, c_K$ are real numbers and $c_K \neq 0$.

Solution of linear homogeneous recurrence relation with constant coefficient

The solution is of the form y_n = Homogeneous soln. + Particular soln.

ie., $y_n = HS + PS$

Rules to find HS:

1. write the characteristic eqn.
2. Solve and find the roots

Roots	HS
1] α_1, α_2 are distinct	$A\alpha_1^n + B\alpha_2^n$
2] α_1, α_2 are equal	$(A+nB)\alpha^n$
3]. $\alpha_1 = \alpha + i\beta$ & $\alpha_2 = \alpha - i\beta$ $\alpha \pm i\beta$	i). $A(\alpha+i\beta)^n + B(\alpha-i\beta)^n$ ii). $r^n(A\cos n\theta + B\sin n\theta)$ where $r = \sqrt{\alpha^2+\beta^2}$ $\theta = \tan^{-1}(\beta/\alpha)$

Rules to find PS:

	$f(n)$	General term
1]	k, a constant	A
2]	k^n, k is a constant	i) Ank^n, if k is a root of characteristic eqn. ii). An^2k^n, if k is a double root of Eqn. iii). Ak^n, if k is not a root of CE.
3]	$f(n)$, a polynomial in n of degree r	$A_0 n^r + A_1 n^{r-1} + \dots + A_n$
4]	$k^n f(n)$ where $f(n)$ is a polynomial in n of degree r and k is a constant.	$(A_0 n^r + A_1 n^{r-1} + \dots + A_n)k^n$

Note:

Order of a recurrence relation

= Highest subscript − Lowest subscript

Eg: $F_n - F_{n-1} - F_{n-2} = 0$

order $= n - (n-2) = 2$

Problems:

1]. If the sequence $a_n = 3 \cdot 2^n$, $n \geq 1$, then find the corresponding recurrence relation.

Givn. $a_n = 3 \cdot 2^n$

$a_{n-1} = 3 \cdot 2^{n-1}$

$= 3 \cdot \frac{2^n}{2}$

$2a_{n-1} = 3 \cdot 2^n$

$= a_n$

$a_n = 2a_{n-1}$, $n \geq 1$ and $a_0 = 3 \cdot 2^0$

$a_0 = 3$

2]. Find the recurrence relation for $S(n) = 6(-5)^n, n \geq 0$

Gvn. $S(n) = 6(-5)^n$

Now $S(n-1) = 6(-5)^{n-1}$

$= \frac{6(-5)^n}{-5}$

$-5\, S(n-1) = S(n)$

$S(n) = -5\, S(n-1), \quad n \geq 0.$

3]. Find the recurrence relation from $y_K = A.\, 2^K + B.\, 3^K$

Givn. $y_K = A.\, 2^K + B.\, 3^K \longrightarrow (1)$

Now $y_{K+1} = A.\, 2^{K+1} + B.\, 3^{K+1}$

$= A.\, 2^K.\, 2 + B.\, 3^K.\, 3$

$= 2A.\, 2^K + 3B.\, 3^K \longrightarrow (2)$

$y_{K+2} = A.\, 2^{K+2} + B.\, 3^{K+2}$

$= 4A\, 2^K + 9B\, 3^K \longrightarrow (3)$

Solving (1), (2) and (3),

$$\begin{vmatrix} y_K & 1 & 1 \\ y_{K+1} & 2 & 3 \\ y_{K+2} & 4 & 9 \end{vmatrix} = 0$$

$y_K [18-12] - 1[9y_{K+1} - 3y_{K+2}] + 1[4y_{K+1} - 2y_{K+2}] = 0$

$6y_K - 9y_{K+1} + 3y_{K+2} + 4y_{K+1} - 2y_{K+2} = 0$

$y_{K+2} - 5y_{K+1} + 6y_K = 0$

4] Find the recurrence relation from $y_n = A3^n + B(-2)^n$

Given $y_n = A\,3^n + B(-2)^n \longrightarrow (1)$

Now, $y_{n+1} = A\,3^{n+1} + B(-2)^{n+1}$

$= 3A\,3^n - 2B(-2)^n \longrightarrow (2)$

$y_{n+2} = A\,3^{n+2} + B(-2)^{n+2}$

$= 9A\,3^n + 4B(-2)^n \longrightarrow (3)$

Solving (1), (2) and (3),

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 3 & -2 \\ y_{n+2} & 9 & 4 \end{vmatrix} = 0$$

$y_n(12+18) - 1(4y_{n+1} + 2y_{n+2}) + 1(9y_{n+1} - 3y_{n+2}) = 0$

$30y_n - 4y_{n+1} - 2y_{n+2} + 9y_{n+1} - 3y_{n+2} = 0$

$-5y_{n+2} + 5y_{n+1} + 30y_n = 0$

$\div(-5)$ $\quad y_{n+2} - y_{n+1} - 6y_n = 0$

5]. Solve $a_n - 7a_{n-1} + 10a_{n-2} = 0$, $n \geq 2$ with $a_0 = 4$, $a_1 = 17$

Given $a_n - 7a_{n-1} + 10a_{n-2} = 0$

characteristic eqn. $m^2 - 7m + 10 = 0$

$(m-2)(m-5) = 0$

$m = 2, 5$

HS $a_n = A(2)^n + B(5)^n$

Since RHS=0, PS=0.

The Soln. is $a_n = A(2)^n + B(5)^n$

Given $a_0 = 4$

$\Rightarrow a_0 = A(2)^0 + B(5)^0$

$4 = A + B \rightarrow (1)$

and $a_1 = 17 \Rightarrow a_1 = A(2)^1 + B(5)^1$

$17 = 2A + 5B \rightarrow (2)$

Solving (1) and (2), $A + B = 4$

$2A + 5B = 17$

$(1)\times 2 \Rightarrow 2A + 2B = 8$

$(-) \quad (-) \quad (-)$

$3B = 9 \Rightarrow B = 3$

$(1) \Rightarrow A = 4 - 3$

$A = 1$

$\therefore a_n = 1(2)^n + 3(5)^n$

2]. Solve the recurrence relation $S(K) = -3S(K-1) - 3S(K-2) - S(K-3)$ with the initial conditions $S(0) = 0$, $S(1) = -2$, $S(2) = -1$.

Gvn. $S(K) + 3S(K-1) + 3S(K-2) + S(K-3) = 0$

characteristic eqn: $m^3 + 3m^2 + 3m + 1 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & 0 & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$m = -1$, $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, -1$

$\therefore m = -1, -1, -1$

$HS = (A + Bn + Cn^2)(-1)^n$

Since $RHS = 0 \Rightarrow PS = 0$

$\therefore S(n) = (A + Bn + Cn^2)(-1)^n$

Given $S(0) = 0$ ie., $S(0) = A = 0 \rightarrow (1)$

$S(1) = -2$ ie., $S(1) = (A + B + C)(-1)^1 = -2$

$A + B + C = 2 \rightarrow (2)$

$S(2) = -1 \Rightarrow S(2) = (A + 2B + 4C)(-1)^2 = -1$

$A + 2B + 4C = -1 \rightarrow (3)$

Solving (1), (2) and (3),

Sub. $A = 0$ in (2) & (3),

$$B + C = 2 \rightarrow (4)$$

$$2B + 4C = -1 \rightarrow (5)$$

$$(4)\times 2 \Rightarrow 2B + 2C = 4$$

$$\underset{(-)}{} \quad \underset{(-)}{} \quad \underset{(-)}{}$$

$$2C = -5$$

$$C = -5/2$$

$$(4) \Rightarrow B = 2 - C = 2 - (-5/2)$$

$$B = 2 + \frac{5}{2}$$

$$B = \frac{9}{2}$$

$$\therefore a_n = \left[\frac{9}{2}n - \frac{5}{2}n^2\right](-1)^n$$

3]. Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$, $n \geq 0$, $a_0 = 1$, $a_1 = 3$.

Gvn. $a_{n+2} - 4a_{n+1} + 4a_n = 0$ Replace $n \rightarrow n-2$

$$\Rightarrow a_n - 4a_{n-1} + 4a_{n-2} = 0$$

characteristic eqn: $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

Since $RHS = 0$

$$\therefore a_n = (A + Bn)\,2^n$$

Gvn $a_0 = 1 \Rightarrow A = 1$

$$a_1 = 3 \Rightarrow (A + B)\,2^1 = 3$$

$$2A + 2B = 3$$

$$2 + 2B = 3 \Rightarrow 2B = 3 - 2 = 1$$

$$B = 1/2$$

$$\therefore a_n = \left(1 + \frac{1}{2}n\right)2^n$$