Solve the recurrence relation for the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, ...

Soln.:

Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, ... satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0$, $f_1 = 1$

ie., $f_n - f_{n-1} - f_{n-2} = 0$

Characteristic eqn.: $m^2 - m - 1 = 0$

$$m = \frac{1 \pm \sqrt{5}}{2}$$

Since RHS = 0 ⇒ PS = 0

∴ The soln. is $f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

Given $f_0 = 0 \Rightarrow f_0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0$

$$0 = A + B$$

$$A + B = 0 \rightarrow (1)$$

and $f_1 = 1 \Rightarrow f_1 = A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 \Rightarrow$

$$A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1 \rightarrow (2)$$

Solving (1) & (2), we get

$$A = \frac{1}{\sqrt{5}} \; ; \; B = \frac{-1}{\sqrt{5}}$$

$$\therefore f_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

2. Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$.

Givn $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

characteristic eqn:

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$\begin{array}{c|cccc} & 1 & -6 & 11 & - \\ 1 & 0 & 1 & -5 & \\ \hline & 1 & -5 & 6 & \end{array}$$

$m = 1, 2, 3$

$m = 1, \quad m^2 - 5m + 6 =$

$(m-2)(m-3) =$

Since RHS $= 0 \Rightarrow PS = 0$

$$\therefore a_n = A(1)^n + B(2)^n + C3^n$$

Givn. $a_0 = 2$

$$A + B + C = 2 \rightarrow (1)$$

$a_1 = 5$

$$A + 2B + 3C = 5 \rightarrow (2)$$

and $a_2 = 15$

$$A + B(2^2) + C(3)^2 = 15$$

$$A + 4B + 9C = 15 \rightarrow (3)$$

Solving (1), (2) and (3),

$$(1) \Rightarrow C = 2 - A - B \rightarrow (4)$$

Sub. (4) in (2),

$$A + 2B + 3(2 - A - B) = 5$$

$$A + 2B + 6 - 3A - 3B = 5$$

$$-2A - B = 5 - 6 = -1$$

$$2A + B = 1 \rightarrow (5)$$

Sub. (4) in (3),

$$A + 4B + 9(2 - A - B) = 15$$

$$A + 4B + 18 - 9A - 9B = 15$$

$$-8A - 5B = -3$$

$$8A + 5B = 3 \rightarrow (6)$$

Solving (5) & (6),

$$(5) \times 4 \Rightarrow 8A + 4B = 4$$

$$(6) \Rightarrow 8A +$$

$$(5) \times 5 \Rightarrow 10A + 5B = 5$$

$$(6) \Rightarrow \underset{(-)}{8A} + \underset{(-)}{5B} = \underset{(-)}{3}$$

$$2A = 2 \Rightarrow A = 1$$

Sub $A = 1$ in (5),

$$2 + B = 1 \Rightarrow B = -1$$

$(1) \Rightarrow A + B + C = 2$

$$1 - 1 + C = 2$$

$$C = 2$$

$\therefore$ Solution is $a_n = 1(1)^n - 1(2)^n + 2(3)^n$

$$a_n = 1^n - 2^n + 2(3)^n$$

6]. Solve the recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$, $n \geq 2$ and $a_0 = 1$, $a_1 = 2$

Givn.

$$a_n - 2a_{n-1} + 2a_{n-2} = 0$$

characteristic eqn.

$$m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= \frac{2[1 \pm i]}{2}$$

$$m = 1 \pm i \quad (\alpha \pm i\beta)$$

$\therefore$ Solution is $a_n = r^n (A \cos n\theta + B \sin n\theta)$

where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}(\beta/\alpha)$

$$r = \sqrt{2} \qquad = \tan^{-1}(1/1) = \tan^{-1}(1)$$

$$\theta = \pi/4$$

$$\therefore a_n = (\sqrt{2})^n \left[A \cos \frac{n\pi}{4} + B \sin \frac{n\pi}{4}\right] \rightarrow (A)$$

Givn $a_0 = 1 \Rightarrow a_0 = A = 1$

and $a_1 = 2 \Rightarrow a_1 = (\sqrt{2})\left[A \cos \frac{\pi}{4} + B \sin \frac{\pi}{4}\right] = 2$

$$A\sqrt{2} \times \frac{1}{\sqrt{2}} + B\sqrt{2} \times \frac{1}{\sqrt{2}} = 2$$

$$A + B = 2$$
$$B = 2 - 1$$
$$B = 1$$

$\therefore$ Solution is $a_n = (\sqrt{2})^n \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right)$

HW 1. Solve the recurrence relation
$s(n) - 10s(n-1) + 9s(n-2) = 0$ with $s(0) = 3, s(1) = 11$

Linear Non-Homogeneous Recurrence Equations:

A recurrence relation of the form
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$
where $c_1, c_2, \ldots, c_k$ are real nos. & $F(n)$ is a fn. not identically zero depending only on n, is called a non homogeneous recurrence relation with constant coefficients.

1. Solve the recurrence relation
$S(k) - 7S(k-1) + 10S(k-2) = 8k+6 \rightarrow (1)$ with $S(0) = 1$, $S(1) = 2$

Given. $S(k) - 7S(k-1) + 10S(k-2) = 0$

CE: $m^2 - 7m + 10 = 0$

$$(m-2)(m-5) = 0$$
$$m = 2, 5$$

$$HS = A(2)^k + B(5)^k \rightarrow (2)$$

PS

RHS = $8k+6$

Take
$$\left.\begin{aligned} S(k) &= ck + d \\ S(k-1) &= c(k-1) + d \\ S(k-2) &= c(k-2) + d \end{aligned}\right\} \rightarrow (A)$$

Subs. (A) in (1),

$$ck+d-7\,[c(k-1)+d]+10\,[c(k-2)+d]=8k+6$$

$$ck+d-7ck+7c-7d+10ck-20c+10d=8k+6$$

$$4ck-13c+4d=8k+6$$

Equating the coefficient of k and constant,

$$4c=8 \qquad -13c+4d=6$$
$$c=2 \qquad -13(2)+4d=6$$
$$4d=6+26$$
$$4d=32$$
$$d=8$$

$$\therefore PS=ck+d$$
$$=2k+8 \rightarrow (3)$$

General soln.

$$S(k)=A(2)^k+B(5)^k+2k+8 \rightarrow (4)$$

Givn. $S(0)=1$

$$S(0)=A+B+8=1$$
$$A+B=-7 \rightarrow (5)$$

and $S(1)=2$

$$S(1)=A(2)+B(5)+2+8=2$$
$$2A+5B=-8 \rightarrow (6)$$

Solving (5) and (6),

$$A+B=-7 \rightarrow (5)$$
$$2A+5B=-8 \rightarrow (6)$$
$$(5)\times 2 \Rightarrow \underset{(-)}{2A}+\underset{(-)}{2B}=\underset{(+)}{-14}$$
$$3B=6 \Rightarrow B=2$$

Subs. $B = 2$ in (5),

$$A + B = -7$$
$$A = -2 - 7 = -9$$
$$A = -9$$

Subs. A & B in (4),

$$S(K) = -9(2)^K + 2(5)^K + 2K + 8$$

3. Solve the Recurrence Relation
$a_n - a_{n-1} - 6a_{n-2} = -30, \; a_0 = 0, \; a_1 = -5, \; n \geq 2$

Givn. $a_n - a_{n-1} - 6a_{n-2} = -30 \rightarrow (1)$

CE: $m^2 - m - 6 = 0$

$$(m-3)(m+2) = 0$$
$$m = 3, -2$$

$$HS = A(3)^n + B(-2)^n \rightarrow (2)$$

PS

RHS = a constant

Take $a_n = a_{n-1} = a_{n-2} = d$

$(1) \Rightarrow \; d - d - 6d = -30$

$$-6d = -30$$
$$d = 5$$

$$PS = 5 \rightarrow (3)$$

General soln.

$$a_n = A(3)^n + B(-2)^n + 5 \rightarrow (4)$$

Givn. $a_0 = 0$

$$A + B + 5 = 0$$
$$A + B = -5 \rightarrow (5)$$

and $a_1 = -5$

$$3A - 2B + 5 = -5$$

$$3A - 2B = -10 \rightarrow (6)$$

Solving (5) and (6),

$$(5) \times 2 \Rightarrow \quad 2A + 2B = -10$$
$$3A - 2B = -10$$
$$5A = -20$$
$$A = -\frac{20}{5} = -4$$

$$(5) \Rightarrow \quad -4 + B = -5$$
$$B = -5 + 4$$
$$B = -1$$

$$(4) \Rightarrow \quad a_n = -4(3)^n - 1(-2)^n + 5$$

7. Solve $a_n - 2a_{n-1} - 3a_{n-2} = 4^n + 6$

Givn. $a_n - 2a_{n-1} - 3a_{n-2} = 4^n + 6 \rightarrow (1)$

E : $m^2 - 2m - 3 = 0$

$$(m-3)(m+1) = 0$$
$$m = 3, -1$$
$$HS = A(3)^n + B(-1)^n \rightarrow (2)$$

PS:
$$RHS = 4^n + 6$$
$$PS = PS_1 + PS_2$$

PS_1: Take
$$\left.\begin{aligned} a_n &= d.4^n \\ a_{n-1} &= d.4^{n-1} \\ a_{n-2} &= d.4^{n-2} \end{aligned}\right\} \rightarrow (3)$$

Subs. (3) in (1),

$$d.4^n - 2d.4^{n-1} - 3d.4^{n-2} = 4^n$$
$$d.4^n - 2d.\frac{4^n}{4} - 3d.\frac{4^n}{16} = 4^n$$

$$4^n\left[d - \frac{d}{2} - \frac{3d}{16}\right] = 4^n$$

$$\frac{16d - 8d - 3d}{16} = 1$$

$$\frac{5d}{16} = 1$$

$$d = \frac{16}{5}$$

$$PS_1 = \frac{16}{5}(4)^n$$

PS_2:

RHS = a constant

Take $a_n = a_{n-1} = a_{n-2} = d$

$$d - 2d - 3d = 6$$

$$-4d = 6$$

$$d = 6/{-4}$$

$$d = \frac{-3}{2}$$

$$PS_2 = \frac{-3}{2} \qquad PS = \frac{16}{5}(4)^n - \frac{3}{2}$$

General soln.

$$a_n = A(3)^n + B(-1)^n + \frac{16}{5}(4)^n - \frac{3}{2}$$

4]. Solve $a_n - 4a_{n-1} + 4a_{n-2} = 2^n + 3n, \ n \geq 2$

Givn. $a_n - 4a_{n-1} + 4a_{n-2} = 2^n + 3n \quad \to (1)$

CE:

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$HS = (A + nB)2^n$$

PS :

$RHS = 2^n + 3n$

$PS = PS_1 + PS_2$

$PS_1 = 2^n$

Take $a_n = dn^2 2^n$ since base of RHS is 2, which is a double root of CE

$a_{n-1} = d(n-1)^2 2^{n-1}$

$a_{n-2} = d(n-2)^2 2^{n-2}$

$\therefore (1) \Rightarrow dn^2 2^n - 4[d(n-1)^2 2^{n-1}] + 4[d(n-2)^2 2^{n-2}] = 2^n$

$\div 2^n \quad dn^2 - 4d(n-1)^2 2^{-1} + 4d(n-2)^2 2^{-2} = 1$

$dn^2 - \frac{4d}{2}(n^2+1-2n) + \frac{4d}{4}(n^2+4-4n) = 1$

$dn^2 - 2d(n^2+1-2n) + d(n^2+4-4n) = 1$

$dn^2 - 2dn^2 - 2d + 4dn + dn^2 + 4d - 4dn = 1$

$2d = 1$

$d = 1/2$

$PS_1 = \frac{1}{2} n^2 (2)^n$

$PS_2 = 3n$

Take $a_n = d_0 + d_1 n$

$a_{n-1} = d_0 + d_1(n-1)$

$a_{n-2} = d_0 + d_1(n-2)$

$(1) \Rightarrow d_0 + d_1 n - 4[d_0 + d_1(n-1)] + 4[d_0 + d_1(n-2)] = 3n$

$d_0 + d_1 n - 4d_0 - 4d_1 n + 4d_1 + 4d_0 + 4d_1 n - 8d_1 = 3n$

$d_0 - 4d_1 + d_1 n = 3n$

Equating the coeffi. of n and constant.

$d_1 = 3; \quad d_0 - 4d_1 = 0$

$d_0 = 4d_1 = 12$

$d_0 = 12$

$PS_2 = 12 + 3n \qquad PS = \frac{1}{2}(n)^2(2)^n + 12 + 3n$

General soln.

$a_n = (A + nB)(2)^n + \frac{1}{2}(n)^2(2)^n + 12 + 3n$

HW 7. Solve $S(K) - 5S(K-1) + 6S(K-2) = 2$ with $S(0) = 1, \ S(1) = -1$.