



DEPARTMENT OF MATHEMATICS

Generating Functions:

The generating function for the sequence 's' with terms a_0, a_1, \dots, a_n of real numbers is the infinite sum.

$$G(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$
$$= \sum_{n=0}^{\infty} a_n x^n$$

Working Rule:

Step 1: Rewrite the recurrence eqn. with RHS = 0

Step 2: Multiply the eqn. obtained in step 1 by x^n & summing from $(0 \text{ to } \infty)$ or $(1 \text{ to } \infty)$ or $(2 \text{ to } \infty)$

Step 3: Put $G(x) = \sum_{n=0}^{\infty} a_n x^n$ and write $G(x)$ as a function of x .

Step 4: Decompose $G(x)$ into partial fraction.

Step 5: Express a_n as the co-efficient of x^n in $G(x)$

Note:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

J. Use the method of generating function to solve the recurrence eqn. $a_n = 3a_{n-1} + 1, n \geq 1$ given $a_0 = 1$.



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$$\text{Given: } a_n = 3a_{n-1} + 1$$

$$\text{Let } G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow a_n - 3a_{n-1} - 1 = 0$$

multiply by x^n & taking summation

$$\sum_{n=1}^{\infty} x^n a_n - 3 \sum_{n=1}^{\infty} x^n a_{n-1} - \sum_{n=1}^{\infty} x^n = 0$$

$$\sum_{n=1}^{\infty} x^n a_n - 3 \sum_{n=1}^{\infty} x \frac{x^n}{x} a_{n-1} - \sum_{n=1}^{\infty} x^n = 0$$

$$\sum_{n=1}^{\infty} x^n a_n - 3x \sum_{n=1}^{\infty} x^{n-1} a_{n-1} - \sum_{n=1}^{\infty} x^n = 0$$

$$[G(x) - a_0] - 3x G(x) - [x + x^2 + x^3 + \dots] = 0$$

$$G(x)[1 - 3x] - a_0 - [(1-x)^{-1} - 1] = 0$$

$$G(x)[1 - 3x] - 1 - (1-x)^{-1} + 1 = 0$$

$$G(x) \cdot (1 - 3x) = \frac{1}{1-x}$$

$$G(x) = \frac{1}{(1-x)(1-3x)} \rightarrow (1)$$

using partial fraction,

$$\frac{1}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$1 = A(1-3x) + B(1-x)$$

$$\text{when } x=1, \quad A = -1/2$$

$$x=0, \quad B = 3/2$$

$$\therefore \frac{1}{(1-x)(1-3x)} = \frac{-1/2}{1-x} + \frac{3/2}{1-3x}$$

$$(1) \Rightarrow G(x) = -\frac{1}{2} \frac{1}{1-x} + \frac{3}{2} \frac{1}{1-3x}$$



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$$= -\frac{1}{2} (1-x)^{-1} + \frac{3}{2} (1-3x)^{-1}$$

$$f(x) = -\frac{1}{2} [1+x+x^2+\dots] + \frac{3}{2} [1+3x+(3x)^2+\dots]$$

$$\therefore a_n x^n = -\frac{1}{2} \sum_{n=0}^{\infty} x^n + \frac{3}{2} \sum_{n=0}^{\infty} (3x)^n$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} (1)^n x^n + \frac{3}{2} \sum_{n=0}^{\infty} (3)^n x^n$$

$$\therefore a_n = -\frac{1}{2} (1)^n + \frac{3}{2} (3)^n \quad [\because a_n - \text{the coeff. of } x^n]$$

Using generating functions, solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$ given that $a_0 = 2, a_1 = 1$

$$\text{Let } G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Given $a_{n+2} - 2a_{n+1} + a_n = 2^n$ with $a_0 = 2, a_1 = 1$

Multiply by x^n and taking summation

$$\sum_{n=0}^{\infty} x^n a_{n+2} - 2 \sum_{n=0}^{\infty} x^n a_{n+1} + \sum_{n=0}^{\infty} x^n a_n = \sum_{n=0}^{\infty} 2^n x^n$$

$$\sum_{n=0}^{\infty} \frac{x^{n+2}}{x^2} a_{n+2} - 2 \sum_{n=0}^{\infty} \frac{x^{n+1}}{x} a_{n+1} + G(x) = \sum_{n=0}^{\infty} (2x)^n$$

$$\frac{1}{x^2} [G(x) - a_0 - a_1 x] - \frac{2}{x} [G(x) - a_0] + G(x) = (1-2x)^{-1}$$

$$\frac{G(x)}{x^2} - \frac{2}{x^2} - \frac{x}{x^2} - \frac{2}{x} G(x) + \frac{4}{x} + G(x) = \frac{1}{1-2x}$$

$$G(x) \left[\frac{1}{x^2} - \frac{2}{x} + 1 \right] + \frac{4}{x} - \frac{2}{x^2} - \frac{1}{x} = \frac{1}{1-2x}$$

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$$G(x) \left[\frac{1-2x+x^2}{x^2} \right] + \frac{2}{x} - \frac{2}{x^2} = \frac{1}{1-2x}$$

$$G(x) \left[\frac{(x-1)^2}{x^2} \right] = \frac{1}{1-2x} + \frac{2-3x}{x^2}$$

$$= \frac{x^2 + (1-2x)(2-3x)}{x^2(1-2x)}$$

$$= \frac{x^2 + 2 - 3x - 4x + 6x^2}{x^2(1-2x)}$$

$$G(x) = \frac{7x^2 - 7x + 2}{x^2(1-2x)} \cdot \frac{x^2}{(x-1)^2}$$

$$G(x) = \frac{7x^2 - 7x + 2}{(1-2x)(x-1)^2}$$

By using partial fractions

$$\frac{7x^2 - 7x + 2}{(1-2x)(x-1)^2} = \frac{A}{1-2x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \rightarrow (1)$$

$$= \frac{A(x-1)^2 + B(1-2x)(x-1) + C(1-2x)}{(1-2x)(x-1)^2}$$

$$7x^2 - 7x + 2 = A(x-1)^2 + B(1-2x)(x-1) + C(1-2x)$$

$$x=1, \quad 2 = C(1-2)$$

$$-C = 2 \Rightarrow C = -2$$

$$x=0, \quad 2 = A - B + C \Rightarrow A - B = 4$$

$$x = \frac{1}{2}, \quad \frac{7}{4} - \frac{7}{2} + 2 = \frac{A}{4} + 0 + 0$$

$$\frac{A}{4} = \frac{7-14+8}{4} = \frac{15-14}{4}$$

$$A = 1$$

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$$\begin{aligned} \therefore B &= A - 4 \\ &= 1 - 4 \end{aligned}$$

$$B = -3$$

Here $A = 1$, $B = -3$, $C = -2$

$$(1) \Rightarrow G(x) = \frac{1}{1-2x} - \frac{3}{x-1} - \frac{2}{(x-1)^2}$$

$$= (1-2x)^{-1} + 3(1-x)^{-1} - 2(1-x)^{-2}$$

$$= 1 + 2x + (2x)^2 + (2x)^3 + \dots + 3[1 + x + x^2 + x^3 + \dots] - 2[1 + 2x + 3x^2 + 4x^3 + \dots]$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (2x)^n + 3 \sum_{n=0}^{\infty} (x)^n - 2 \sum_{n=0}^{\infty} (n+1)x^n$$

$$a_n = (2)^n + 3(1)^n - 2(n+1)$$

3. using generating function, solve the recurrence relation $a_{n+2} - 5a_{n+1} - 6a_n = 2$, by using $a_0 = 1$, $a_1 = 2$.

4. Solve the recurrence relation, $a_{n+2} - 4a_{n+1} + 3a_n = 0$, $a_0 = 2$, $a_1 = 4$ using generating function.

5. Solve $a_n - 7a_{n-1} + 10a_{n-2} = 0$, $a_0 = 10$, $a_1 = 41$, using generating function.