PUZZLE-5

Solving riddles with boolean algebra

ou're on an island with knights and knaves. All knights tell the truth and all knaves lie. All the inhabitants know who is a knight and who is a knave. You meet Gildas, Ergard and Telones on the island. Gildas says: "We're not all knights." Ergard says: "We're not all knaves." Telones says: "I'm the only knight." What are they?

And I was wondering how one would go about solving this riddle with boolean algebra. Can it be done with a truth table?

Solution:
The key to using boolean algebra on this kind of problem is that a person is a knight if and only if what they say is true. That is, you'll get an equality "X is a knight" = "What X says is true". If you can encode both these things algebraically, you'll have a set of three equations to be solved.

I like using boolean rings for this kind of problem. In a boolean ring, multiplication is interpreted as "and" and addition is interpreted as "exclusive or" ("xor"). 0 is "false" and 1 is "true". It follows from that that all the usual operations on a ring work as usual. You have distribution of multiplication over addition, you have additive inverses, and 0 and 1 work as the additive and multiplicative identities respectively.

An important operation (for this problem) is negation. We'll be interpreting "not x" as (1 + x). You can check that "not X = true xor X" in propositional logic.

On top of being a ring, boolean rings are nontrivial ($0 \neq 1$), commutative ($x * y = y * x$), characteristic 2 ($x + x = 0$) and idempotent ($x * x = x$).

Alright, let's start. We'll be interpreting "Gildas is a knight" as $g = 1$, and similarly for the others.

Gildas says: "We're not all knights."

This can be interpreted as g = 1 + get. get is the product of g, e, and t, so is the interpretation of "Gildas, Ergard and Telones are all knights". Adding 1 negates this to "Gildas, Ergard and Telones are not all knights". Finally, this is true if and only if Gildas is a knight, so we get g = 1 + get.

Ergard says: "We're not all knaves."

This one is very similar. "x is a knave" means (x = 0), or (x + 1 = 1). So our interpretation will be e = 1 + (1 + g)(1 + e)(1 + t).

Telones says: "I'm the only knight."

This one's easier. This means that t is a knight, but neither of the others are. t = (1 + g)(1 + e)t.

Now we need to solve this system of equations. Since it's non-linear, you may need to get creative (though look up Gröbner bases for a more principled approach). I started with the simplest equation: g = 1 + get. Note that if et = 1, then this would simplify to g = 1 + g, which has no solutions (it's equivalent to 0 = 1). So we must have et = 0, so g = 1 + g * 0 = 1. Gildas is a knight.

From there, the problem quickly falls apart. The second equation becomes e = 1 + (1 + 1)(1 + e)(1 + t). Recalling that 1 + 1 = 0 (characteristic 2), this means that e = 1. Ergard is a knight.

The third equation reduces to t = (1 + 1)(1 + 1)t = 0 * 0 * t = 0. Telones is a knave.

In fact, we don't even need that third equation. Remember that the first equation gave us that et = 0. Since we know that e = 1 from the second equation, we get t = 0 from that. Telones didn't need to say anything for us to know that he was a knave.