GRAPH TERMINOLOGY:

DEGREE OF A VERTEX:

The number of edges incident at the vertex v_i is called the degree of the vertex with self loops counted twice and it is denoted by $d(v_i)$.

IN-DEGREE AND OUT-DEGREE OF A DIRECTED GRAPH.

In a directed graph, the in-degree of a vertex v, denoted by $deg^-(v)$ and defined by the number of edges with v as their terminal vertex.

The out-degree of v, denoted by $deg^+(v)$, is the number of edges with v as their initial vertex.

THEOREM: 1

THE HANDSHAKING THEOREM:

Let $G=(V,E)$ be an undirected graph with 'e' edges.

Then $\sum_{v\in V} deg(v) = 2e$. The sum of degrees of all the vertices of an undirected graph is twice the number of edges of the graph and hence even.

Proof:

Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

$\therefore$ All the 'e' edges contribute (2e) to the sum of the degrees of vertices.

$\therefore \sum \deg(v) = 2e$.

THEOREM: 2

In a undirected graph, the number of odd degree vertices are even.

Proof:

Let V_1 and V_2 be the set of all vertices of even degree and set of all vertices of odd degree, respectively, in a graph $G=(V,E)$

$$\therefore \sum d(v) = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j)$$

By Handshaking theorem, we have

$$2e = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j) \quad \text{———} (1)$$

Since each $\deg(v_i)$ is even, $\sum_{v_i \in V_1} \deg(v_i)$ is even.

from eqn. (1). As LHS is 2e, is even and in RHS first expression is even, then second expression must be even.

$$\therefore \sum_{v_j \in V_2} d(v_j) \text{ is even.}$$

Since each deg (v_j) is odd, the number of terms contained in $\sum_{v_j \in V_2} d(v_j)$ must be even.

$\therefore$ The no. of vertices of odd degree is even.

[Hint: The max. no. of degree with 'n' vertices of a graph G is $n-1$.]

THEOREM 3:

The maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.

Proof: Let $G=(V,E)$ be a simple graph with 'n' vertices.

By handshaking theorem, $\sum_{i=1}^{n} \deg(v_i) = 2e$

where 'e' is the no. of edges with 'n' vertices in the graph G.

(ie) $\deg(v_1) + \deg(v_2) + \deg(v_3) + \cdots + \deg(v_n) = 2e$

Since we know that the maximum degree of each vertex in the graph G can be $n-1$.

$$\therefore (n-1) + (n-1) + \cdots + (n-1) = 2e$$

$$n(n-1) = 2e$$

$$\therefore \frac{n(n-1)}{2} = e$$

Hence, the maximum no. of edges in any simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.