



UNIT 1 PARTIAL DIFFERENTIAL EQUATIONS

Linear partial differential equations of second order with constant coefficients of homogeneous types

Type: 2 RHS =  $\cos(ax+by)$  or  $\sin(ax+by)$

Replace  $D^2 \rightarrow -a^2$

$DD' \rightarrow -ab$

$D'^2 \rightarrow -b^2$

1. Solve!  $(D^2 - 2DD' + D'^2)z = \cos(x-3y)$

Auxiliary Equation is  $m^2 - 2m + 1 = 0$

$$(m-1)(m-1) = 0$$

$$m = 1, 1 \text{ (equal)}$$

$\therefore$  CF is  $f_1(y+x) + xf_2(y+x)$

$$PI = \frac{1}{D^2 - 2DD' + D'^2} \cos(x-3y)$$

$$= \frac{1}{-1 - 2(3) - 9} \cos(x-3y)$$

$$= \frac{-1}{16} \cos(x-3y)$$

$$a=1, b=-3$$

$$D^2 \rightarrow -a^2 = -1$$

$$DD' = -ab = -(1)(-3) = 3$$

$$D'^2 = -b^2 = -(-3)^2 = -9$$

$$z = CF + PI$$

$$= f_1(y+x) + xf_2(y+x) + \frac{1}{16} \cos(x-3y)$$

2. Solve!  $(D^2 - 4D'^2)z = \sin(2x+y)$

AE is  $m^2 - 4 = 0 \Rightarrow m^2 = 2^2 \quad m = \pm 2$   
 $m = 2, -2$  (Different)

$$a=2$$

$$b=1$$

$$D^2 = -a^2 = -2^2 = -4$$

$$CF = f_1(y-2x) + f_2(y+2x)$$

$$PI = \frac{1}{D^2 - 4D'^2} \sin(2x+y) = \frac{1}{-4 - 4(1)} \sin(2x+y)$$

$$= \frac{x \sin(2x+y)}{2D} = \frac{x \sin(2x+y)}{-2(2)} = \frac{-x \sin(2x+y)}{4}$$

$$DD' \rightarrow -ab = -2$$

$$D'^2 = -b^2 = -1^2 = -1$$

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$$PI = \frac{-x}{4} \cos(2x+y)$$

The solution is  $z = CF + PI$

$$= f_1(y-2x) + f_2(y+2x) - \frac{x}{4} \cos(2x+y)$$

3. Find the PI of  $(D^2 - 3DD' + D'^2)z = \sin x \cos y$ .

$$PI = \frac{1}{D^2 - 3DD' + D'^2} \sin x \cos y$$

$$\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$$

$$\text{Given: } (D^2 - 3DD' + D'^2)z = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$$

$$= \frac{1}{2} \left[ \frac{1}{D^2 - 3DD' + D'^2} \sin(x+y) \right.$$

$$\left. + \frac{1}{D^2 - 3DD' + D'^2} \sin(x-y) \right]$$

$$= \frac{1}{2} [PI_1 + PI_2] \rightarrow \text{①}$$

$$PI_1 = \frac{1}{D^2 - 3DD' + D'^2} \sin(x+y)$$

$$a=1, b=1$$

$$D^2 \Rightarrow -a^2 = -1$$

$$DD' \Rightarrow -ab = -1$$

$$D'^2 \Rightarrow -b^2 = -1$$

$$= \frac{1}{-1 - 3(-1) + (-1)} \sin(x+y)$$

$$= \frac{1}{-1 + 3 - 1} \sin(x+y) = \sin(x+y)$$

$$PI_2 = \frac{1}{D^2 - 3DD' + D'^2} \sin(x-y) = \frac{1}{-1 - 3(1) - 1} \sin(x-y)$$

$$= -\frac{1}{5} \sin(x-y)$$

$$PI = \frac{1}{2} \left[ \sin(x+y) - \frac{1}{5} \sin(x-y) \right]$$

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Linear partial differential equations of second order with constant coefficients of homogeneous types

H.W: 1. Find the PI of  $(D^2 + 3DD' - 4D'^2)z = \sin y$

2. Find the PI of  $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = \sin(3x+2y)$

Type 3:  $x^m y^n$

1. Solve  $(D^2 - 4DD' + 4D'^2)z = xy$

AE is  $m^2 - 4m + 4 = 0 \Rightarrow (m-2)(m-2) = 0$   
 $m = 2, 2.$

CF =  $f_1(y+2x) + x f_2(y+2x)$

$$PI = \frac{1}{D^2 - 4DD' + 4D'^2} xy = \frac{1}{D^2 \left[ 1 - \frac{4DD'}{D^2} + \frac{4D'^2}{D^2} \right]} xy$$

$$= \frac{1}{D^2} \left[ 1 - \left( \frac{4D'}{D} - \frac{4D'^2}{D^2} \right) \right]^{-1} xy$$

$$= \frac{1}{D^2} \left[ 1 + \left( \frac{4D'}{D} - \frac{4D'^2}{D^2} \right) + \dots \right] xy \quad \left[ \because (1-x)^{-1} = 1+x+x^2+\dots \right]$$

$$= \frac{1}{D^2} \left[ xy + \frac{4D'}{D}(xy) - 0 \right]$$

$$= \frac{1}{D^2} \left[ xy + \frac{4}{D}x \right] = \frac{1}{D^2} xy + \frac{4}{D^3} x$$

$$\left[ \begin{array}{l} \frac{1}{D^2} xy \xrightarrow{1st} \frac{1}{D} \frac{x^2}{2} y \xrightarrow{2nd} \frac{x^3}{6} y \\ \frac{1}{D^3} x \xrightarrow{1st} \frac{1}{D^2} \frac{x^2}{2} \xrightarrow{2nd} \frac{1}{D} \frac{x^3}{6} \rightarrow \frac{x^4}{24} \end{array} \right]$$

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Linear partial differential equations of second order with constant coefficients of homogeneous types

$$= \frac{x^3 y + 4x^4}{6}$$

$$= \frac{x^3 y}{6} + \frac{x^4}{6}$$

∴ The solution is  $z = CF + PI$

$$z = f_1(y+2x) + x f_2(y+2x) + \frac{1}{6} [x^3 y + x^4]$$

2) Find the PI of  $(D^2 - DD' - 2D'^2)z = 2x + 3y$

$$PI = \frac{1}{D^2 - DD' - 2D'^2} (2x + 3y)$$

$$= \frac{1}{D^2 \left[ 1 - \frac{DD'}{D^2} - \frac{2D'^2}{D^2} \right]} (2x + 3y) = \frac{1}{D^2} \left[ 1 - \left( \frac{D'}{D} - \frac{2D'^2}{D^2} \right) \right]^{-1} (2x + 3y)$$

$$= \frac{1}{D^2} \left[ 1 + \left( \frac{D'}{D} - \frac{2D'^2}{D^2} \right) + \dots \right] (2x + 3y)$$

$$= \frac{1}{D^2} \left[ 2x + 3y + \frac{D'}{D} (2x + 3y) \right]$$

$$= \frac{1}{D^2} \left[ 2x + 3y + \frac{1}{D} (3) \right] = \frac{1}{D^2} (2x + 3y) + \frac{1}{D^3} (3)$$

$$\frac{1}{D^2} (2x + 3y) \rightarrow \frac{1}{D} \left( \frac{2x^2}{2} + 3xy \right) \rightarrow \frac{x^3}{3} + \frac{3x^2 y}{2}$$

$$\frac{1}{D^3} (3) \rightarrow \frac{1}{D^2} 3x \rightarrow \frac{1}{D} \frac{3x^2}{2} \rightarrow \frac{x^3}{2}$$

$$PI = \frac{x^3}{3} + \frac{3x^2 y}{2} + \frac{x^3}{2}$$

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Linear partial differential equations of second order with constant coefficients of homogeneous types

$$\text{RHS} \Rightarrow e^{ax+by} + \sin(ax+by) \text{ (or) } e^{ax+by} + \cos(ax+by)$$

1) Solve:  $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x-y)$

AE is  $m^2 - m - 20 = 0$   
 $(m-5)(m+4) = 0 \Rightarrow m = 5, -4$

CF =  $f_1(y-4x) + f_2(y+5x)$

$$\text{PI} = \frac{1}{D^2 - DD' - 20D'^2} [e^{5x+y} + \sin(4x-y)]$$

$$= \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y} + \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$$

=  $\text{PI}_1 + \text{PI}_2$

$\text{PI}_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y}$

$D \rightarrow a = 5$   
 $D' \rightarrow b = 1$

$$= \frac{1}{25 - 5 - 20} e^{5x+y}$$

$$= \frac{x}{2D - D} e^{5x+y} = \frac{x}{2(5) - 1} e^{5x+y}$$

$$= \frac{x}{9} e^{5x+y} \quad \text{PI}_1 = \frac{x}{9} e^{5x+y}$$

$\text{PI}_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$

$a = 4 \quad b = -1$   
 $D^2 \rightarrow a^2 = -16$   
 $DD' \rightarrow -ab = +4$   
 $D'^2 \rightarrow -b^2 = -(-1)^2 = -1$

$$= \frac{1}{-16 - 4 - 20(-1)} \sin(4x-y)$$

$$= x \frac{1}{2D - D'} \sin(4x-y)$$

$$= x \frac{2D + D'}{(2D)^2 - D'^2} \sin(4x-y) = \frac{x [2D \sin(4x-y) + D'(\sin(4x-y))]}{-64 + 1}$$

$$= \frac{-x}{63} [8 \cos(4x-y) - \cos(4x-y)]$$

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Linear partial differential equations of second order with constant coefficients of homogeneous types

$$= \frac{-7x}{63} \cos(4x-y)$$

$$= \frac{-x}{9} \cos(4x-y)$$

The soln is  $z = CF + PI$

$$= f_1(y-4x) + f_2(y+5x) + \frac{x}{9} e^{5x+y} - \frac{x}{9} \cos(4x-y)$$

HW: 1.  $(D^2 + 4DD' - 5D'^2)z = e^{2x-y} + \sin(x-2y)$

2.  $(D^2 - DD' - 30D'^2)z = xy + e^{6x+y}$

3.  $r + s - 6t = y \cos x$

Type-IV RHS =  $f(xy) = e^{ax+by} x^m y^n$  (or)  $e^{ax+by} \cos(ax+by)$   
 $\sin(ax+by)$

$$PI = \frac{1}{\phi(D, D')} e^{ax+by} x^m y^n$$

Replace  $D \rightarrow D+a$ ;  $D' \rightarrow D'+b$  then III rule or type II rule

1. Solve!  $(D^2 - 2DD' + D'^2)z = x^2 y^2 e^{x+y}$

AE  $\& \ m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0 \quad m = 1$$

$\therefore$  The roots are real and equal

CF  $\& \ z = f_1(y+mx) + x f_2(y+mx)$

$$= f_1(y+x) + x f_2(y+x)$$

$$PI_{\&} = \frac{1}{D^2 - 2DD' + D'^2} x^2 y^2 e^{x+y}$$

Replace  $D \rightarrow D+1$ ,  $D' \rightarrow D'+1$

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$$\begin{aligned}
 &= \frac{1}{(D+1)^2 - 2(D+1)(D'+1) + (D'+1)^2} e^{x+y} x^2 y^2 \\
 &= \frac{1}{D^2 + 2D + 1 - 2DD' - 2D - 2D' - 2 + D'^2 + 2D' + 1} e^{x+y} x^2 y^2 \\
 &= \frac{1}{D^2 - 2DD' + D'^2} e^{x+y} x^2 y^2 \\
 &= e^{x+y} \frac{1}{D^2} \left[ 1 - \left( \frac{2D'}{D} - \frac{D'^2}{D^2} \right) \right]^{-1} x^2 y^2 \\
 &= e^{x+y} \frac{1}{D^2} \left[ 1 + \left( \frac{2D'}{D} - \frac{D'^2}{D^2} \right) + \dots \right] x^2 y^2 \\
 &= e^{x+y} \frac{1}{D^2} \left[ x^2 y^2 + \frac{2D'}{D} x^2 y^2 - \frac{D'^2}{D^2} x^2 y^2 \right] \\
 &= e^{x+y} \left[ \frac{1}{D^2} x^2 y^2 + \frac{2D'}{D^3} x^2 y^2 - \frac{D'^2}{D^4} x^2 y^2 \right] \\
 &= e^{x+y} \left[ \frac{1}{D^2} x^2 y^2 + \frac{4}{D^3} x^2 y - \frac{2}{D^4} x^2 \right] \\
 &= e^{x+y} \left[ \frac{x^2 y^2}{12} + \frac{4x^2 y}{60} - \frac{2x^2}{360} \right] \\
 &= e^{x+y} \left[ \frac{x^2 y^2}{12} + \frac{x^2 y}{15} - \frac{x^2}{180} \right]
 \end{aligned}$$

∴ The soln is  $z = CF + PI$

$$= f_1(y+x) + 2f_2(y+x)$$

$$+ e^{x+y} \left[ \frac{x^2 y^2}{12} + \frac{x^2 y}{15} - \frac{x^2}{180} \right]$$