Unit - III

Partial Differential Equation :

A Differential Equation which depends on more than one independent variable, is called partial Differential Equation.

For eg.,

1). $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} + z = 0$

2). $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y^2$

Order :

The order of the PDE is the highest partial derivative which occurs in it.

Degree :

The degree of the PDE is the power of the highest partial derivative which occur in it.

For eg.,

$$\frac{\partial^3 z}{\partial x^3} + 4 \frac{\partial^2 z}{\partial x^2} + 6 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + y^2$$

Order : 3

Degree : 1

Formation of PDE :

* Elimination of Arbitrary constants
* Elimination of Arbitrary functions

Formation of PDE by Elimination of Arbitrary constants:

Notations:

$$p = \frac{\partial z}{\partial x} \qquad r = \frac{\partial^2 z}{\partial x^2}$$

$$q = \frac{\partial z}{\partial y} \qquad s = \frac{\partial^2 z}{\partial x \partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

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1. Form the PDE from $z = ax + by + \sqrt{a^2+b^2}$

Soln.:

Given $z = ax + by + \sqrt{a^2+b^2} \rightarrow (1)$

Differentiate partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = a + 0 + 0$$

$$\Rightarrow p = a \rightarrow (2)$$

Differentiate partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = 0 + b + 0$$

$$\Rightarrow q = b \rightarrow (3)$$

Sub. (2) and (3) in (1),

$$z = px + qy + \sqrt{p^2+q^2}$$

2]. Form the PDE from $ax^2 + by^2 + z^2 = 1$

Soln.:

Given $ax^2 + by^2 + z^2 = 1 \rightarrow (1)$

Differentiate partially w.r.t 'x'

$$2ax + 0 + 2z\frac{\partial z}{\partial x} = 0$$

$$2ax = -2zp$$

$$a = \frac{-zp}{x}$$

Differentiate partially w.r.t 'y'

$$0 + 2by + 2z\frac{\partial z}{\partial y} = 0$$

$$2by + 2zq = 0$$

$$2by = -2zq$$

$$b = \frac{-zq}{y}$$

Subs. a and b in (1),

$$-\frac{zp}{x}x^2 - \frac{zq}{y}y^2 + z^2 = 1$$

$$-zpx - zqy + z^2 = 1$$

$$z(z - px - qy) = 1$$

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3). Form the PDE by eliminating a and b from $z = (x^2+a^2)(y^2+b^2)$

Soln. :

Given. $z = (x^2+a^2)(y^2+b^2) \rightarrow (1)$

Differentiate partially w.r.t 'x'

$$p = 2x(y^2+b^2)$$

$$y^2+b^2 = \frac{p}{2x} \rightarrow (2)$$

Differentiate partially w.r.t 'y'

$$q = 2y(x^2+a^2)$$

$$x^2+a^2 = \frac{q}{2y} \rightarrow (3)$$

Subs. (2) & (3) in (1), $z = \frac{q}{2y}\frac{p}{2x}$

$$pq = 4xyz$$

4]. Form the PDE $(x-a)^2+(y-b)^2+z^2=1$

Soln. :

Given $(x-a)^2+(y-b)^2+z^2=1 \rightarrow (1)$

$$\Rightarrow 2(x-a)+2zp=0$$

$\div 2$ $$x-a=-zp \rightarrow (2)$$

$$\Rightarrow 2(y-b)+2zq=0$$

$\div 2$ $$y-b=-zq \rightarrow (3)$$

Subs. (2) and (3) in (1),

$$(-zp)^2+(-zq)^2+z^2=1$$

$$z^2p^2+z^2q^2+z^2=1$$

$$z^2(p^2+q^2+1)=1$$

5]. Form the PDE of the family of spheres having their centres on the line $x=y=z$.

Soln. :

centre (a,b,c) lie on $x=y=z$ i.e., $a=b=c$

The required eqn. of the sphere,

$$(x-a)^2+(y-a)^2+(z-a)^2=r^2 \rightarrow (1)$$

Diff. w.r.t 'x'

$$2(x-a)+2(z-a)p=0$$

$\div$ $$(x-a)+(z-a)p=0$$

$$x+zp-a(1+p)=0 \Rightarrow a(1+p)=x+zp$$

$$a=\frac{x+zp}{1+p} \rightarrow (2)$$

$$\Rightarrow 2(y-a)+2(z-a)q=0$$

$\div 2$ $$y-a+(z-a)q=0$$

$$y+zq-a-aq=0 \Rightarrow a=\frac{y+zq}{1+q} \rightarrow (3)$$

From (2) and (3), $\dfrac{x+zp}{1+p} = \dfrac{y+zq}{1+q}$

$$\Rightarrow p(z-y)+q(x-z)=y-x$$

6. Form the PDE from $\log(az-1) = x + ay + b$

soln. :

Differentiate partially w.r.to 'x'

$$\frac{1}{az-1} ap = 1 \rightarrow (1)$$

Differentiate partially w.r.to 'y'

$$\frac{1}{az-1} aq = a$$

$$q = az-1 \Rightarrow q+1 = az \Rightarrow a = \frac{q+1}{z} \rightarrow (2)$$

subs. q in (1),

$$\frac{ap}{q} = 1 \Rightarrow a = \frac{q}{p} \rightarrow (3)$$

from (2) and (3), $\frac{q}{p} = \frac{q+1}{z}$

$$qz = pq + p$$

$$qz - pq - p = 0$$

$$p + pq - qz = 0$$

7. Form the PDE from $z = ax + by + cxy \rightarrow (1)$

soln. :

$$p = a + cy$$

$$q = b + cx$$

Number of Arbitrary constants (3) > Number of ~~Arbitrary~~ Indept. variables (2)

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(a+cy) = 0$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(b+cx) = c$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(b+cx) = 0$$

Subs. $s = c$ in p and q

$$p = a + sy \quad | \quad q = b + sx$$
$$a = p - sy \quad | \quad b = q - sx$$

$$z = (p - sy)x + (q - sx)y + sxy$$
$$= px - syx + qy - sxy + sxy$$
$$z = px + qy - syx$$

HW

I. Form the PDE from i). $z = ax + by + ab$

ii). $z = (x+a)^2 + (y-b)^2$

iii). $z = (2x^2+a)(3y-b)$

Formation of PDE by Elimination of Arbitrary functions:

I. Form the PDE by eliminating the arbitrary functions from

i). $z = f(x^2+y^2+z^2)$

ii). $z = x^2 + 2g\left(\frac{1}{y} + \log x\right)$

iii) $f(xy+z^2, x+y+z) = 0$

iv). $z = f(x+t) + g(x-t)$

Soln. :

i). $z = f(x^2+y^2+z^2) \rightarrow (1)$

Differentiate partially w.r.t 'x'

$$p = f'(x^2+y^2+z^2)(2x + 2zp)$$

$$f'(x^2+y^2+z^2) = \frac{p}{2x+2zp} \rightarrow (2)$$

Differentiate partially w.r.t 'y'

$$q = f'(x^2+y^2+z^2)(2y+2zq)$$

$$f'(x^2+y^2+z^2) = \frac{q}{2y+2zq} \rightarrow (3)$$

From (2) and (3),

$$\frac{p}{2x+2zp} = \frac{q}{2y+2zq}$$

$$p(2y+2zq) = q(2x+2zp)$$

$$2yp+2zpq = 2xq+2zpq$$

$\div 2$ $\quad yp = xq$

$$py = qx$$

ii). $z = x^2+2g\left(\frac{1}{y}+\log x\right) \rightarrow (1)$

Differentiate partially w.r.t 'x'

$$p = 2x+2g'\left(\frac{1}{y}+\log x\right)\frac{1}{x}$$

$$p-2x = 2g'\left(\frac{1}{y}+\log x\right)\frac{1}{x}$$

$$\Rightarrow g'\left(\frac{1}{y}+\log x\right) = \frac{x}{2}(p-2x) \rightarrow (2)$$

Differentiate partially w.r.t 'y'

$$q = 2g'\left(\frac{1}{y}+\log x\right)\left(\frac{-1}{y^2}\right)$$

$$g'\left(\frac{1}{y}+\log x\right) = -\frac{y^2}{2}q \rightarrow (3)$$

From (2) and (3),

$$\frac{x}{2}(p-2x) = -\frac{y^2}{2}q$$

$(\times 2)$ $\quad xp-2x^2 = -y^2q$

$$px-2x^2+qy^2 = 0$$

$$px+qy^2 = 2x^2$$

iii). $f(xy+z^2,\ x+y+z)=0$

Here $u = xy+z^2$ | $v = x+y+z$

$u_x = y+2zp$ | $v_x = 1+p$

$u_y = x+2zq$ | $v_y = 1+q$

Now,

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = 0$$

$$\begin{vmatrix} y+2zp & 1+p \\ x+2zq & 1+q \end{vmatrix} = 0$$

$$(y+2zp)(1+q) - (1+p)(x+2zq) = 0$$

$$y+yq+2zp+2zpq - x-2zq-px-2zpq = 0$$

$$p(2z-x)+q(y-2z) = x-y$$

iv). $z = f(x+t)+g(x-t)$

$$p = f'(x+t)+g'(x-t)$$

$$q = f'(x+t)-g'(x-t)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+t)+g''(x-t) \longrightarrow (1)$$

$$s = \frac{\partial^2 z}{\partial x \partial t} = f''(x+t)-g''(x-t) \longrightarrow (2)$$

$$t = \frac{\partial^2 z}{\partial t^2} = f''(x+t)+g''(x-t) \longrightarrow (3)$$

From (1) and (3), $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial t^2}$

HW 1. $z = f\left(\dfrac{xy}{z}\right)$

2. $\phi\left(z^2-xy-\dfrac{x}{z}\right)=0$