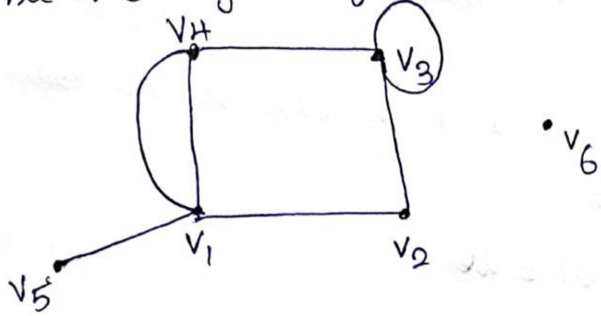
Graph Terminology

Degree of a vertex:

The number of edges incident at the vertex V_i is called the degree of the vertex with self loops counted twice and it is denoted by $d(V_i)$.

I). Find the degree of the vertices for the graph



$d(V_1) = 4$ $\quad$ $d(V_4) = 3$

$d(V_2) = 2$ $\quad$ $d(V_5) = 1$

$d(V_3) = 4$ $\quad$ $d(V_6) = 0$

Indegree and Outdegree of a directed graph:

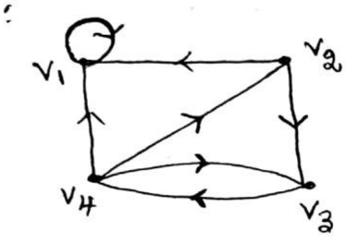
In a directed graph, the in-degree of a vertex V, denoted by $deg^-(V)$ and defined by the number of edges with V as their terminal vertex.

The out-degree of V, denoted by $deg^+(V)$, is the number of edges with V as their initial vertex.

Note:

A loop at a vertex contributes 1 to both in & out degree

Eg:



Indegree	Outdegree
$d^-(v_1) = 3$	$d^+(v_1) = 1$
$d^-(v_2) = 1$	$d^+(v_2) = 2$
$d^-(v_3) = 2$	$d^+(v_3) = 1$
$d^-(v_4) = 1$	$d^+(v_4) = 3$

Theorem 1: (Handshaking Theorem)

Let $G = (V, E)$ be an undirected graph with 'e' edges. Then $\sum_{v \in V} \deg(v) = 2e$.

The sum of degrees of all the vertices of an undirected graph is twice the number of edges of the graph and hence even.

Proof:

Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

$\therefore$ All the 'e' edges contribute $(2e)$ to the sum of the degrees of vertices.

$$\therefore \sum \deg(v) = 2e$$

Theorem 2:

In a undirected graph, the number of odd degree vertices are even.

Proof:

Let V_1 and V_2 be the set of all vertices of even degree and odd degree respectively, in a graph G.

$$\therefore \sum d(v) = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j) \rightarrow (1)$$

By the handshaking theorem,

$$2e = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j) \rightarrow (2)$$

In (2), LHS is even and the first expression on the RHS is even, we have the 2nd expression on the RHS must be even.

i.e., $\sum_{v_j \in V_2} d(v_j)$ is even.

Since each deg (v_j) is odd, the number of terms contained in $\sum_{v_j \in V_2} d(v_j)$ must be even.

$\therefore$ The number of vertices of odd degree is even.

Theorem 3:

The maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$

Proof:

We prove this theorem by the principle of mathematical Induction.

Let P(n) be the maximum no. of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

For $n=1$, $P(1): \frac{1(1-1)}{2} = 0$

$\therefore$ A graph with one vertex has no edges.

$\therefore$ P(1) is true

Assume that P(K) be the maximum no. of edges in a simple graph with K vertices is $\frac{K(K-1)}{2}$.

To prove P(K+1) is true.

Let G be a graph having K+1 vertices and G' be the graph obtained from G by deleting 1 vertex i.e., $v \in V(G)$

By our assumption, G' has atmost $\frac{K(K-1)}{2}$ edges.

Now we add the vertex v to G' such that it may be adjacent to all the k vertices of G'

$$\text{Total no. of edges} = \frac{k(k-1)}{2} + k$$

$$= \frac{k(k-1)+2k}{2}$$

$$= \frac{k^2-k+2k}{2}$$

$$= \frac{k(k+1)}{2}$$

$$= \frac{(k+1)(k+1-1)}{2}$$

$\therefore$ The result is true for $k+1$ vertices.

Hence the maximum no. of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Problems

1). How many edges are there in a graph with 10 vertices each of degree '6'.

Soln.

Let the no. of edges be 'e'

Given. No. of vertices $= 10$

Degree of each vertex $= 6$

By handshaking thm.,

$$\Sigma \deg(v) = 2e$$

$$6 \times 10 = 2e$$

$$2e = 60$$

$$\boxed{e = 30}$$

2). How many vertices does a regular graph with degree '4' with '10' edges have?

Soln. Let the no. of vertices be 'v'.

Givn. No. of edges = 10

degree of each vertex = 4

By handshaking thm., $\Sigma \deg(v) = 2e$

$4 \times v = 2(10)$

$\boxed{v = 5}$

3] Can a simple graph exist with '15' vertices each of degree '3'

Soln.

Let the no. of edges be e

Givn. No. of vertices = 15

Degree of each vertex = 3

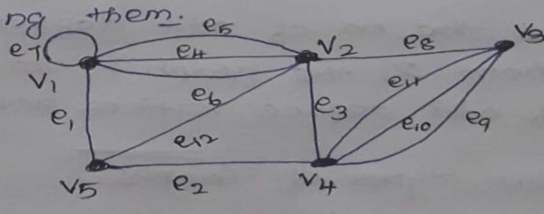
By handshaking thm., $\Sigma \deg(v) = 2e$

$3 \times 15 = 2e$

$e = \frac{45}{2}$ which is not an integer.

$\therefore$ A simple graph cannot exist.

4] Find the no. of vertices, no. of edges and degree of each vertex in the following undirected graph and also verify handshaking thm.



Soln.

No. of vertices : 5

No. of edges : 12

$d(v_1) = 6$ | $d(v_4) = 5$

$d(v_2) = 6$ | $d(v_5) = 3$

$d(v_3) = 4$

$\therefore$ Total degree $= 6+6+4+5+3$

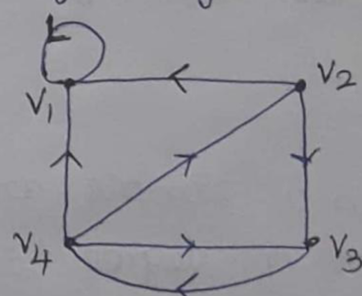
By handshaking thm. $\Sigma d(v) = 2e$

$24 = 2(12)$

$= 24$

Hence proved.

5]. Find the indegree of the directed graph & also outdegree of the directed graph. S.T no. of edges is equal to the total no. of indegree.



Soln.

Indegree	Out degree
$d^-(v_1) = 3$	$d^+(v_1) = 0$
$d^-(v_2) = 1$	$d^+(v_2) = 2$
$d^-(v_3) = 2$	$d^+(v_3) = 1$
$d^-(v_4) = 1$	$d^+(v_4) = 3$

$\therefore$ Total No. of indegree $= 7$

and No. of edges $= 7$

$\therefore$ No. of edges $=$ Total No. of indegree

Hence proved.

6]. Is there any graph with degree sequence $(1, 3, 3, 3, 5, 6, 6)$?

Soln.

Here the no. of odd degree vertices $= 5$ $(1, 3, 3, 3, 5)$

By thm 2, the graph is not possible since the no. of odd degree vertices are even.