UNIT 2 FOURIER SERIES
PARSEVAL'S IDENTITY

Parseval's Identity

For the interval $(-l, l)$, the parseval's Identity is

$$\frac{1}{2l}\int_{-l}^{l}[f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

For the Interval $(0, 2l)$

$$\frac{1}{l}\int_{0}^{2l}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

For half range cosine series,

$$\frac{2}{l}\int_{0}^{l}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

For half range sine series

$$\frac{2}{l}\int_{0}^{l}[f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2.$$

1. Find the fourier series of $f(x) = x^2$ in $-\pi < x < \pi$

and deduce that

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

iv) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$

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Put $x=0$ in ①

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos 0$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{-\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

$$\frac{-\pi^2}{12} = \left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots\right].$$

$$\frac{-\pi^2}{12} = -\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\right]$$

$$\frac{\pi^2}{12} = \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\right] \rightarrow ②$$

Put $x=\pi$ in ①

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos n\pi$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos n\pi$$

$$\frac{2\pi^2}{3} = 4\left[\frac{(-1)\cos\pi}{1^2} + \frac{(-1)^2\cos 2\pi}{2^2} + \frac{(-1)^3\cos 3\pi}{3^2} + \dots\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{(-1)(-1)}{1^2} + \frac{(1)(1)}{2^2} + \frac{(-1)(-1)}{3^2} + \dots\right]$$

$$\frac{\cancel{2}\pi^2}{3} = \cancel{4}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

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$f(x) = x^2$ in $-\pi < x < \pi$

$f(x)$ is even function

Fourier series is

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$\boxed{a_0 = \frac{2}{3}\pi^2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$u = x^2$ $\quad v = \cos nx$

$u' = 2x$ $\quad v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\quad v_2 = -\frac{\cos nx}{n^2}$

$v_3 = \frac{-\sin nx}{n^3}$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2 \frac{\sin n\pi}{n} + 2\pi\frac{\cos n\pi}{n^2} - 2\frac{\sin n\pi}{n^3} - \left[0 + 0 - \frac{2\sin 0}{n^3}\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right] = \frac{4(-1)^n}{n^2}. \quad \boxed{a_n = \frac{4(-1)^n}{n^2}.}$$

$\therefore$ The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\frac{2}{3}\pi^2}{2} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n}\frac{(-1)^n}{n^2}\cos nx \longrightarrow (1)$$

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$$\frac{\pi^2}{6} = \left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\therefore \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6} \rightarrow (3)$$

Adding (2) & (3)

$$2\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] = \frac{\pi^2}{12}+\frac{\pi^2}{6} = \frac{\pi^2+2\pi^2}{12}$$

$$= \frac{3\pi^2}{12}$$

$$2\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] = \frac{\pi^2}{4}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$$

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2} \text{ \& } b_n = 0.$$

$$\int_{-\pi}^{\pi} (x^2)^2\,dx = 2\pi\left[\frac{4\pi^4}{9.4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16(-1)^{2n}}{n^4}\right]$$

$$\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{2\pi^5}{9} + 16\pi\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^4}$$

$$\frac{2\pi^5}{5} = \frac{2\pi^5}{9} + 16\pi\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]$$

$$\frac{2\pi^5}{5} - \frac{2\pi^5}{9} = 16\pi\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]$$

$$\frac{8\pi^5}{45} = 16\pi\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]$$

$$\frac{\pi^4}{90} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$$