UNIT 2 FOURIER SERIES
GENERAL FOURIER SERIES

General Fourier Series

If $f(x)$ is a periodic function and satisfies Dirichlet's condition defined for the interval $[c, c+2l]$ then it can be represented by an infinite series is called Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where a_0, a_n and b_n are called Fourier coefficients.

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

1. Find the Fourier series for the function $f(x) = x^2$ in $(0, 2\pi)$

$f(x) = x^2$ in $(0, 2\pi)$

Fourier series for the function $f(x)$ in $[0, 2\pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad [\text{Put } l = \pi]$$

UNIT 2 FOURIER SERIES
GENERAL FOURIER SERIES

To find a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\,dx \qquad = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2\pi)^3}{3} - 0\right] \qquad = \frac{1}{\cancel{\pi}}\left(\frac{8\pi^{\cancel{3}}}{3}\right)$$

$$= \frac{8\pi^2}{3} \qquad \boxed{a_0 = \frac{8\pi^2}{3}}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$u = x^2$ $\qquad v = \cos nx$

$u' = 2x$ $\qquad v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\qquad v_2 = \frac{-\cos nx}{n^2}$

$u''' = 0$ $\qquad v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{1}{\pi}\left[uv_1 - u'v_2 + u''v_3\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + 2(2\pi)\frac{\cos n2\pi}{n^2} - 0 - 0 - 0 + 0\right]$$

(1)

$$= \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right) = \frac{4}{n^2}.$$

$$a_n = \frac{4}{n^2}$$

UNIT 2 FOURIER SERIES
GENERAL FOURIER SERIES

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$u = x^2$ $\quad v = \sin nx$

$u' = 2x$ $\quad v_1 = -\frac{\cos nx}{n}$

$u'' = 2$ $\quad v_2 = -\frac{\sin nx}{n^2}$

$u''' = 0$ $\quad v_3 = \frac{\cos nx}{n^3}$

$$= \frac{1}{\pi}\left[uv_1 - u'v_2 + u''v_3\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2\cos n2\pi}{n} + 2(2\pi)\frac{\sin n2\pi}{n^2} + 2\frac{\cos n2\pi}{n^3} + 0 - 0 - \frac{2\cos 0}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + 0 + \frac{2}{n^3} - \frac{2}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2}{n}\right]$$

$$b_n = \frac{-4\pi}{n}$$

$$\therefore f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{4\pi}{n}\sin nx.$$

(2) Find the fourier series for the function

$f(x) = \frac{(\pi - x)^2}{2}$ in $0 \le x \le 2\pi$

$f(x) = \frac{(\pi - x)^2}{2}$

Fourier series for the function $f(x)$ in the interval $[0, 2\pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{2}\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi-x)^2\,dx = \frac{1}{2\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi}$$

$$= \frac{-1}{6\pi}\left[(\pi-2\pi)^3 - \pi^3\right] = \frac{-1}{6\pi}\left[(-\pi)^3 - \pi^3\right]$$

$$= \frac{2\pi^3}{6\pi} = \frac{\pi^2}{3}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{2}\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi-x)^2 \cos nx\,dx$$

$u = (\pi - x)^2$ $\qquad$ $v = \cos nx$

$u' = 2(\pi - x)(-1)$ $\qquad$ $v_1 = \frac{\sin nx}{n}$

$= -2(\pi - x)$ $\qquad$ $v_2 = \frac{-\cos nx}{n^2}$

$u'' = -2(-1) = 2,\ u''' = 0$ $\qquad$ $v_3 = \frac{-\sin nx}{n^3}$

$$a_n = \frac{1}{2\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - [-2(\pi-x)]\left[\frac{-\cos nx}{n^2}\right] + 2\left[\frac{-\sin nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[0 - 2(-\pi)\frac{\cos n(2\pi)}{n^2} - 0 - 0 + 2(\pi)\frac{\cos 0}{n^2} + 0\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{2\pi}\left[\frac{4\pi}{n^2}\right]$$

$$\boxed{a_n = \frac{2}{n^2}}$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{2}\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

$u = (\pi-x)^2$ $\quad\quad v = \sin nx$

$u' = 2(\pi-x)(-1)$ $\quad\quad v_1 = \frac{-\cos nx}{n}$

$= -2(\pi-x)$ $\quad\quad v_2 = \frac{-\sin nx}{n^2}$

$u'' = -2(-1) = 2$ $\quad\quad v_3 = \frac{\cos nx}{n^3}.$

$u''' = 0$

$$b_n = \frac{1}{2\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - \left[-2(\pi-x)\left(-\frac{\sin nx}{n^2}\right)\right] + 2\left[\frac{\cos nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-(\pi-2\pi)^2\frac{\cos n(2\pi)}{n} - 0 + \frac{2\cos n2\pi}{n^3} + \frac{\pi^2\cos 0}{n} + 0 - \frac{2\cos 0}{n^3}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right] \quad \boxed{b_n = 0}$$

The fourier series is

$$f(x) = \frac{\pi^2/3}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2} \cos nx + 0$$

$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2} \cos nx$$

Interval : $[0, 2l]$

③ Find the fourier series for the function $f(x) = x^2$ in $(0, 2l)$

$$f(x) = x^2 \text{ in } (0, 2l)$$

The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

To find a_0:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\int_0^{2l} x^2\,dx$$

$$= \frac{1}{l}\left[\frac{x^3}{3}\right]_0^{2l} = \frac{1}{3l}\left[8l^3 - 0\right]$$

$$a_0 = \frac{8l^2}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^{2l} x^2 \cos\left(\frac{n\pi x}{l}\right)dx$$

UNIT 2 FOURIER SERIES
GENERAL FOURIER SERIES

$u = x^2$ $\qquad v = \cos\left(\frac{n\pi x}{l}\right)$

$u' = 2x$ $\qquad v_1 = \dfrac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}$

$u'' = 2$

$u''' = 0$ $\qquad v_2 = \dfrac{-\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}$

$\qquad v_3 = \dfrac{-\sin(n\pi x/l)}{(n\pi/l)^3}$

$$a_n = \frac{1}{l}\left[x^2\left(\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right) - 2x\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right] + 2\left[\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]\right]_0^{2l}$$

$$= \frac{1}{l}\left[0 + 2(2l)\frac{\cos\left(\frac{n\pi 2l}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} - 0 - 0 - 0 + 0\right]$$

$$= \frac{1}{l}\left[4l \cos(n2\pi)\frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{4l^2}{n^2\pi^2}(1)$$

$$a_n = \frac{4l^2}{n^2\pi^2}$$

To find b_n:

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\int_0^{2l} x^2 \sin\left(\frac{n\pi x}{l}\right)dx$$

UNIT 2 FOURIER SERIES
GENERAL FOURIER SERIES

$u = x^2 \qquad v = \sin\left(\frac{n\pi x}{l}\right)$

$u' = 2x \qquad v_1 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}$

$u'' = 2 \qquad v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$

$u''' = 0 \qquad v_3 = \frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}$

$$b_n = \frac{1}{l}\left[\left[-(x^2)\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right] - (2x)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right) + 2\left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[-4l^2\frac{\cos\left(\frac{n\pi 2l}{l}\right)}{\left(\frac{n\pi}{l}\right)} + 2(2l)\left[\frac{\sin\left(\frac{n\pi 2l}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right] + 2\left[\frac{\cos\left(\frac{n\pi 2l}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]\right.$$

$$\left. + 0 - 0 - 2\frac{\cos 0}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{1}{l}\left[\frac{-4l^3}{n\pi}(\cos n2\pi) + \frac{2l^3}{n^3\pi^3}(\cos n2\pi) - \frac{2l^3}{n^3\pi^3}(1)\right]$$

$$= \frac{1}{l}\left[\frac{-4l^3}{n\pi} + \cancel{\frac{2l^3}{n^3\pi^3}} - \cancel{\frac{2l^3}{n^3\pi^3}}\right]$$

$$\boxed{b_n = \frac{-4l^2}{n\pi}}$$

UNIT 2 FOURIER SERIES
GENERAL FOURIER SERIES

∴ The fourier series is

$$f(x) = \frac{\frac{8l^2}{3}}{2} + \sum_{n=1}^{\infty} \left(\frac{4l^2}{n^2\pi^2}\right) \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} \left(\frac{-4l^2}{n\pi}\right) \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{4l^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4l^2}{n^2\pi^2}\right) \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} \left(\frac{-4l^2}{n\pi}\right) \sin\left(\frac{n\pi x}{l}\right)$$