Recurrence Relation

Let $\{a_n\}$ be a sequence of real numbers with a_n as the n^{th} term.

A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses interms of one or more earlier terms ie, $a_1, a_2, \ldots a_{n-1}$ for all integers n with $n \geq n_0$

Eg: fibonacci series : 0, 1, 1, 2, 3, 5, 8, 13, ... which can be represented by the recurrence relation

$f_n = f_{n-1} + f_{n-2}$, $n \geq 2$ with $f_0 = 0$, $f_1 = 1$

Homogeneous Recurrence Relation

A linear homogeneous recurrence relation of degree K with constant coefficients is a recurrence relation of the form, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$, where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$.

Solution of linear homogeneous recurrence relation with constant coefficient

The solution is of the form y_n = Homogeneous soln. + Particular soln.

ie., $y_n = HS + PS$

Rules to find HS:

1. write the characteristic eqn.
2. Solve and find the roots

Roots	HS
1] α_1, α_2 are distinct	$A\alpha_1^n + B\alpha_2^n$
2] α_1, α_2 are equal	$(A+nB)\alpha^n$
3]. $\alpha_1 = \alpha + i\beta$ is $\alpha_2 = \alpha - i\beta$ $\alpha \pm i\beta$	i). $A(\alpha+i\beta)^n + B(\alpha-i\beta)^n$ ii). $r^n(A\cos n\theta + B\sin n\theta)$ where $r = \sqrt{\alpha^2+\beta^2}$, $\theta = \tan^{-1}(\beta/\alpha)$

Rules to find PS:

	$f(n)$	General term
1.	k, a constant	A
2.	k^n, k is a constant	i). Ank^n, if k is a root of characteristic eqn. ii). An^2k^n, if k is a double root of Eqn. iii). Ak^n, if k is not a root of CE.
3.	$f(n)$, a polynomial in n of degree r	$A_0 n^r + A_1 n^{r-1} + \dots + A_n$
4.	$k^n f(n)$ where $f(n)$ is a polynomial in n of degree r and k is a constant.	$(A_0 n^r + A_1 n^{r-1} + \dots + A_n)k^n$

Note:

Order of a recurrence relation

= Highest subscript – lowest subscript

Eg: $F_n - F_{n-1} - F_{n-2} = 0$

$$\text{order} = n - (n-2) = 2$$

Problems:

1. If the sequence $a_n = 3 \cdot 2^n$, $n \geq 1$, then find the corresponding recurrence relation.

Gvn. $a_n = 3 \cdot 2^n$

$$a_{n-1} = 3 \cdot 2^{n-1}$$
$$= 3 \cdot \frac{2^n}{2}$$
$$2a_{n-1} = 3 \cdot 2^n$$
$$= a_n$$
$$a_n = 2a_{n-1}, \quad n \geq 1 \text{ and } a_0 = 3 \cdot 2^0$$
$$a_0 = 3$$

2]. Find the recurrence relation for

$S(n) = 6(-5)^n, \; n \geq 0$

Givn. $S(n) = 6(-5)^n$

Now $S(n-1) = 6(-5)^{n-1}$

$= \dfrac{6(-5)^n}{-5}$

$-5\,S(n-1) = S(n)$

$S(n) = -5\,S(n-1), \; n \geq 0.$

3]. Find the recurrence relation from

$y_k = A \cdot 2^k + B \cdot 3^k$

Givn. $y_k = A \cdot 2^k + B \cdot 3^k \longrightarrow (1)$

Now $y_{k+1} = A \cdot 2^{k+1} + B \cdot 3^{k+1}$

$= A \cdot 2^k \cdot 2 + B \cdot 3^k \cdot 3$

$= 2A \cdot 2^k + 3B \cdot 3^k \longrightarrow (2)$

$y_{k+2} = A \cdot 2^{k+2} + B \cdot 3^{k+2}$

$= 4A\,2^k + 9B\,3^k \longrightarrow (3)$

Solving (1), (2) and (3),

$$\begin{vmatrix} y_k & 1 & 1 \\ y_{k+1} & 2 & 3 \\ y_{k+2} & 4 & 9 \end{vmatrix} = 0$$

$y_k[18-12] - 1[9y_{k+1} - 3y_{k+2}] + 1[4y_{k+1} - 2y_{k+2}] = 0$

$6y_k - 9y_{k+1} + 3y_{k+2} + 4y_{k+1} - 2y_{k+2} = 0$

$y_{k+2} - 5y_{k+1} + 6y_k = 0$

4] Find the recurrence relation from

$$y_n = A3^n + B(-2)^n$$

Given $y_n = A\,3^n + B(-2)^n \longrightarrow (1)$

Now, $y_{n+1} = A\,3^{n+1} + B(-2)^{n+1}$

$= 3A\,3^n - 2B(-2)^n \longrightarrow (2)$

$y_{n+2} = A\,3^{n+2} + B(-2)^{n+2}$

$= 9A\,3^n + 4B(-2)^n \longrightarrow (3)$

Solving (1), (2) and (3),

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 3 & -2 \\ y_{n+2} & 9 & 4 \end{vmatrix} = 0$$

$y_n(12+18) - 1(4y_{n+1} + 2y_{n+2}) + 1(9y_{n+1} - 3y_{n+2}) = 0$

$30y_n - 4y_{n+1} - 2y_{n+2} + 9y_{n+1} - 3y_{n+2} = 0$

$-5y_{n+2} + 5y_{n+1} + 30y_n = 0$

$\div(-5)$ $y_{n+2} - y_{n+1} - 6y_n = 0$