METHOD OF MULTIPLIERS:

Choose any three multipliers (l, m, n) which may be constants or functions of x, y, z such that,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = k.$$

By direct integration, $u(x, y, z) = c_1$

Similarly, choose another multiplier,

$$\frac{l'\,dx + m'\,dy + n'\,dz}{l'P + m'Q + n'R} = s.$$

By direct integration $v(x, y, z) = c_2$

∴ General soln. is $\phi(u, v) = 0$

① $x(y-z)p + y(z-x)q = z(x-y)$

$P = x(y-z)$; $Q = y(z-x)$; $R = z(x-y)$

General form: $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

choose the first set of multiplies $(l, m, n) = (1, 1, 1)$

$$\frac{1 \cdot dx + 1 \cdot dy + 1 \cdot dz}{1(x(y-z)) + 1 \cdot y(z-x) + 1 \cdot z(x-y)} = k.$$

$$\frac{1\cdot dx + 1\cdot dy + 1\cdot dz}{xy - xz + yz - yx + zx - yz} = k$$

$Dr = xy - xz + yz - yx + zx - yz = 0$

$\Rightarrow Nr: \; dx + dy + dz = 0$

$\Rightarrow \; x + y + z = c_1$

$\Rightarrow u(x, y, z) = x + y + z = c_1$.

Choose another set of multiplies $(l', m', n') = (1/x, 1/y, 1/z)$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}\cdot x(y-z) + \frac{1}{y}\cdot y(z-x) + \frac{1}{z}\cdot z(x-y)} = k$$

$\Rightarrow Dr = 0$

$\Rightarrow Nr: \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\Rightarrow \log x + \log y + \log z = \log c_2$

$\Rightarrow v(x, y, z) = xyz = c_2$.

$\therefore$ General soln. is $\phi(x + y + z, xyz) = 0$.

② Solve: $(mz-ny)\frac{\partial z}{\partial x}+(nx-lz)\frac{\partial z}{\partial y}=(ly-mx)$

$\Rightarrow (mz-ny)p+(nx-lz)q=(ly-mx)$

$P=mz-ny\ ;\ Q=nx-lz\ ;\ R=ly-mx$

$$\frac{dx}{mz-ny}=\frac{Q}{nx-lz}=\frac{R}{ly-mx}$$

Choose first set of multipliers $(l,m,n)=(l,m,n)$

$$\frac{l\,dx+m\,dy+n\,dz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}=k$$

Dr: $lmz-lny+mnx-mlz+nly-mnx=0$

Nr: $l\,dx+m\,dy+n\,dz=0$

$\Rightarrow lx+my+nz=c_1 \Rightarrow u(x,y,z)$

Choose the another set of multiplier $(l',m',n')=(x,y,z)$

$$\frac{x\,dx+y\,dy+z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=k$$

Dr: $=0$

Nr $= x\,dx+y\,dy+z\,dz=0$

$\Rightarrow v(x,y,z)=x^2+y^2+z^2=c_2$

General soln is $\phi(lx+my+nz,\ x^2+y^2+z^2)=0$