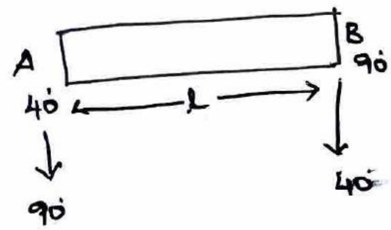
UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
ONE DIMENSIONAL EQUATION OF HEAT CONDUCTION

Type - 2 !

1. The ends A and B of a rod of length 'l' have 40°c and 90°c until steady state prevails. The temp at A is suddenly raised to 90°c and at the same time that at B is lowered to 40°c. find the temperature distribution in the rod at time 't'.

one dimensional Heat Equation is,

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.$$



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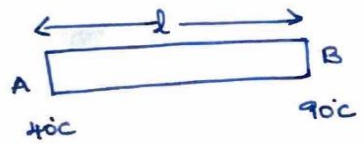
Before Change

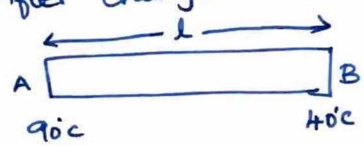


$$u(x) = \left(\frac{b-a}{l}\right)x + a$$

$$= \left(\frac{90-40}{l}\right)x + 40$$

$$= \frac{50x}{l} + 40$$

After Change



$$f(x) = \left(\frac{40-90}{l}\right)x + 90$$

$$= \frac{-50x}{l} + 90$$

The initial temperature is

$$u(x,0) = \frac{50x}{l} + 40$$

The boundary conditions are

i) $u(0,t) = 90$

ii) $u(l,t) = 40$

iii) $u(x,0) = \frac{50x}{l} + 40$

Since the BC are non zero, we assume that

$$u(x,t) = f(x) + v(x,t) \rightarrow (1)$$

$$\Rightarrow v(x,t) = u(x,t) - f(x)$$

a) $v(0,t) = u(0,t) - f(x)$

$$= 90 - 90 = 0$$

b) $v(l,t) = u(l,t) - f(l)$

$$= 40 - \left(\frac{-50l}{l} + 90\right) = 40 + 50 - 90$$

$$= 0$$

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c) $v(x,0) = u(x,0) - f(x)$

$$= \frac{50x}{l} + 40 - \left(-\frac{50x}{l} + 90\right)$$

$$= \frac{100x}{l} - 50.$$

The new boundary conditions are.

a) $v(0,t) = 0$

b) $v(l,t) = 0$

c) $v(x,0) = \frac{100x}{l} - 50.$

The suitable solution is

$$v(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-a^2\lambda^2 t} \longrightarrow ②$$

ⓐ $v(0,t) = 0.$

$A\, e^{-a^2\lambda^2 t} = 0.$

$e^{-a^2\lambda^2 t} \neq 0$ [∵ It is a function of time]

$\boxed{A = 0}$

$② \Rightarrow v(x,t) = B\sin\lambda x\, e^{-a^2\lambda^2 t} \longrightarrow ③$

$ⓑ \Rightarrow v(l,t) = 0$

$B\sin\lambda l\, e^{-a^2\lambda^2 t} = 0.$

$e^{-a^2\lambda^2 t} \neq 0$ [∵ It is a function of time]

$B \neq 0$ [If $B=0$, we get trivial solution]

$\sin\lambda l = 0$

$\lambda l = \sin^{-1} 0$

$\lambda l = n\pi$

$\boxed{\lambda = \frac{n\pi}{l}}$

$③ \Rightarrow v(x,t) = B\sin\frac{n\pi x}{l}\, e^{-\frac{a^2 n^2\pi^2 t}{l^2}} \longrightarrow ④$

The most general solution is

$$V(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \, e^{\frac{-a^2 n^2 \pi^2 t}{l^2}} \rightarrow (5)$$

$$(c) \Rightarrow v(x,0) = \frac{100x}{l} - 50.$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{100x}{l} - 50.$$

By Half Range Sine series,

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \frac{100x}{l} - 50.$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_0^l \left(\frac{100x}{l} - 50\right) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\left(\frac{100x}{l} - 50\right)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - \left(\frac{100}{l}\right)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l}{n\pi}\left[\frac{100l}{l} - 50\right]\left(-\cos\frac{n\pi l}{l}\right) + \left(\frac{100}{l}\right)\left(\frac{l^2}{n^2\pi^2}\right)\left(\sin\frac{n\pi l}{l}\right) - \left\{(-50)\frac{-\cos 0}{n\pi}(l) - \frac{100}{l}\left(\frac{-\sin 0}{n^2\pi^2/l^2}\right)\right\}\right]$$

$$= \frac{2}{l}\left[\frac{-50l}{n\pi}(-1)^n + \cancel{100}\, 0 - \frac{50l}{n\pi} - 0\right]$$

$$= \frac{-100}{n\pi}\left[1 + (-1)^n\right]$$

$$B_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{-200}{n\pi} & \text{if } n \text{ is even} \end{cases}$$

(5) $\Rightarrow$ $$V(x,t) = \sum_{n=\text{even}}^{\infty} \frac{-200}{n\pi} \sin\frac{n\pi x}{l} \, e^{\frac{-a^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = -\frac{50x}{l} + 90 - \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{l} \, e^{\frac{-a^2 n^2 \pi^2 t}{l^2}}$$

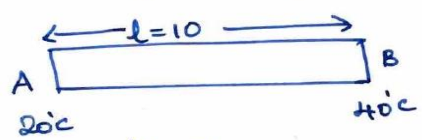
2) A bar of 10cm long with insulated sides, has its ends A and B kept at 20°c and 40°c respectively until steady state conditions prevails. The temperature at A is then suddenly raised to 50°c and at the same instant that at B is lowered to 10°c. Find the subsequent temperature at any point of the bar at any time.

The one dimensional Heat equation is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

First we find the temperature function $u(x,t)$ at any distance before and after the changes of temperature at the ends of the rod. So there are two steady state solutions.

Before change

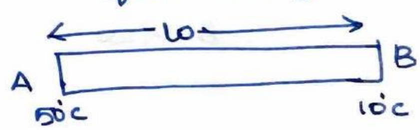


$$u(x) = \left(\frac{b-a}{l}\right)x + a$$

$$= \left(\frac{40-20}{10}\right)x + 20$$

$$= 2x + 20$$

After change.



$$f(x) = \left(\frac{b-a}{l}\right)x + a$$

$$= \left(\frac{10-50}{10}\right)x + 50$$

$$= \frac{-40}{10}x + 50$$

$$= -4x + 50.$$

The initial temperature is

$$u(x,0) = \frac{20x}{10} + 20. = 2x + 20$$

The boundary conditions are

i) $u(0,t) = 50$

ii) $u(10,t) = 10$

iii) $u(x,0) = 2x + 20$

Since the boundary conditions are non zero.

We assume that

$$u(x,t) = f(x) + v(x,t) \rightarrow ①$$

$$v(x,t) = u(x,t) - f(x)$$

a) $v(0,t) = u(0,t) - f(0)$

$= 50 - 50 = 0.$

b) $v(10,t) = u(10,t) - f(10)$

$= 10 - 10 = 0.$

c) $v(x,0) = u(x,0) - f(x) = 2x + 20 - (-4x + 50)$

$= 2x + 20 + 4x - 50$

$= 6x - 30$

The New boundary conditions are

i) $v(0,t) = 0$

ii) $v(10,t) = 0$

iii) $v(x,0) = 6x - 30.$

The suitable solution is,

$$v(x,t) = (A\cos\lambda x + B\sin\lambda x)\,e^{-a^2\lambda^2 t} \longrightarrow ②$$

Apply (i) in ②

$\Rightarrow v(0,t) = 0$

$\Rightarrow [A(1) + B(0)]\,e^{-a^2\lambda^2 t} = 0.$

$A\,e^{-a^2\lambda^2 t} = 0.$

$e^{-a^2\lambda^2 t} \neq 0$ [$\because$ It is a fn of 't']

$\boxed{A = 0}$

Apply $\boxed{A=0}$ in ①

$$v(x,t) = B\sin\lambda x\, e^{-a^2\lambda^2 t} \longrightarrow ③$$

Apply ii) in ③

$v(10,t) = 0.$

$\Rightarrow B\sin\lambda(10)\,e^{-a^2\lambda^2 t} = 0$

$e^{-a^2\lambda^2 t} \neq 0$ [$\because$ It is a fn of 't']

$B \neq 0$ [If $B = 0$, we get trivial solution]

$\therefore \sin\lambda(10) = 0$

$\lambda 10 = \sin^{-1} 0 \qquad 10\lambda = n\pi$

$\boxed{\lambda = \frac{n\pi}{10}}$

Sub λ value in ③

$$v(x,t) = B\sin\frac{n\pi x}{10}\, e^{-\frac{a^2 n^2\pi^2 t}{100}} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}\, e^{-\frac{a^2 n^2 \pi^2 t}{100}} \longrightarrow ④$$

Apply iii) in ④

$v(x,0) = 6x - 30$

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} = 6x - 30.$$

By Half range sine series,

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{10}\int_0^{10}(6x-30)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{1}{5}\int_0^{10}(6x-30)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{1}{5}\left[(6x-30)\left(\frac{-\cos\frac{n\pi x}{10}}{n\pi/10}\right) - (6)\left(-\frac{\sin\frac{n\pi x}{10}}{n^2\pi^2/100}\right)\right]_0^{10}$$

$$= \frac{1}{5}\left[(60-30)\left(\frac{10}{n\pi}\right)\left(-\cos\frac{n\pi 10}{10}\right) - 6\left(-\sin\frac{n\pi 10}{10}\right)\left(\frac{100}{n^2\pi^2}\right)\right.$$

$$\left. - \left\{(6(0)-30)-\cos\frac{n\pi(0)}{10}\cdot\left(\frac{10}{n\pi}\right) - 6\left(\sin\frac{n\pi(0)}{10}\right)\left(\frac{100}{n^2\pi^2}\right)\right\}\right]$$

$$= \frac{1}{5}\left[\frac{-300}{n\pi}(-1)^n - \frac{300}{n\pi}(1)\right]$$

$$= \frac{-300}{5n\pi}\left[1+(-1)^n\right]$$

$$b_n = \frac{-60}{n\pi}\left[1+(-1)^n\right]$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{-120}{n\pi} & \text{if } n \text{ is even.}\end{cases}$$

Sub b_n in (4)

$$v(x,t) = \sum_{n=\text{even}}^{\infty} \frac{-120}{n\pi}\sin\frac{n\pi x}{10}\, e^{\frac{-a^2n^2\pi^2 t}{100}}$$

$$= \frac{-120}{\pi} \sum_{n=\text{even}}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{10}\, e^{\frac{-a^2 n^2 \pi^2 t}{100}}$$

Sub $v(x,t)$ in ①

$$u(x,t) = f(x) + v(x,t)$$

$$= -4x + 50 - \frac{120}{\pi} \sum_{n=\text{even}}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{10}\, e^{\frac{-a^2 n^2 \pi^2 t}{100}}.$$