UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
STEADY STATE SOLUTION OF TWO-DIMENSIONAL EQUATION OF HEAT CONDUCTION(INSULATED EDGES EXCLUDED)

Two Dimensional Heat Equation

Introduction:

The differential equation for two dimensional heat flow for the unsteady case is,

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

In steady state, u is independent of t, ie $\frac{\partial u}{\partial t} = 0$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad [\text{Laplace equation}]$$

Possible solutions:

i) $u(x,y) = (A_1 e^{px} + A_2 e^{-px})(A_3 \cos py + A_4 \sin py)$

ii) $u(x,y) = (A_5 \cos px + A_6 \sin px)(A_7 e^{py} + A_8 e^{-py})$

iii) $u(x,y) = (A_9 x + A_{10})(A_{11} y + A_{12})$

Suitable solution:-

i) If heat flows in x-direction, then

$u(x,y) = (A e^{px} + B e^{-px})(C \cos py + D \sin py)$

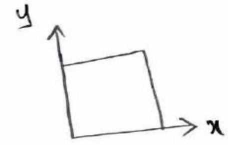
ii) If heat flows in y-direction, then

$u(x,y) = (A \cos px + B \sin px)(C e^{py} + D e^{-py})$

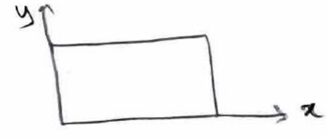
Types of plates:

1) Finite Plates:

i) Square plate



ii) Rectangular plate



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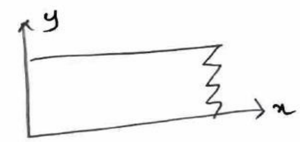
2) Infinite plates:

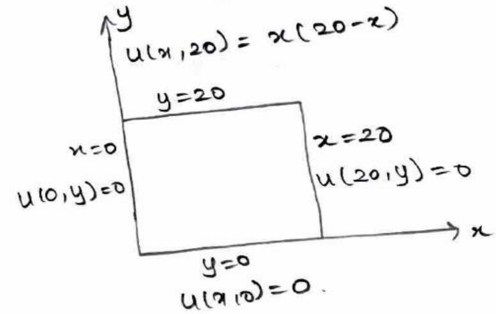
i) Vertically Infinite plate

ii) Horizontally Infinite plate



Finite Plates:

1. The square plate bounded by the line $x=0, y=0, x=20, y=20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,20) = x(20-x)$ while the other three edges are kept at $0°c$. Find the steady state temperature in the plate.



The laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The boundary conditions are

i) $u(0,y)=0$

(ii) $u(20,y)=0$

iii) $u(x,0)=0$

iv) $u(x,20) = x(20-x)$

The suitable solution is

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \rightarrow (1)$$

i) $\Rightarrow u(0,y)=0$

$$(A(1) + B(0))(Ce^{\lambda y} + De^{-\lambda y}) = 0$$

$$A(Ce^{\lambda y} + De^{-\lambda y}) = 0$$

Here $Ce^{\lambda y} + De^{-\lambda y} \neq 0$ [$\because$ It is a function of 'y']

$$\Rightarrow A = 0$$

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$(1) \Rightarrow u(x,y) = B \sin \lambda x \, (C e^{\lambda y} + D e^{-\lambda y}) \rightarrow (2)$

$ii) \Rightarrow u(20, y) = 0$

$B \sin 20\lambda \, (C e^{\lambda y} + D e^{-\lambda y}) = 0$

Here $C e^{\lambda y} + D e^{-\lambda y} \neq 0$ and $B \neq 0$

$\Rightarrow \sin 20\lambda = 0$

$20\lambda = n\pi \quad \lambda = \frac{n\pi}{20}$

$(2) \Rightarrow u(x,y) = B \sin \frac{n\pi x}{20} \left(C e^{\frac{n\pi y}{20}} + D e^{-\frac{n\pi y}{20}} \right) \rightarrow (3)$

$iii) \Rightarrow u(x, 0) = 0$

$B \sin \frac{n\pi x}{20} \, [C + D] = 0$

Here $B \neq 0$, $\sin \frac{n\pi x}{20} \neq 0 \Rightarrow C + D = 0 \Rightarrow \boxed{D = -C}$

$(3) \Rightarrow u(x,y) = B \sin \frac{n\pi x}{20} \left[C e^{\frac{n\pi y}{20}} - C e^{-\frac{n\pi y}{20}} \right]$

$= BC \sin \frac{n\pi x}{20} \left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}} \right]$

$= 2BC \sin \frac{n\pi x}{20} \sinh \frac{n\pi y}{20}$

$= B \sin \frac{n\pi x}{20} \sinh \frac{n\pi y}{20}$

$\sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2}$

where $2BC = B$