Linear Non-Homogeneous Recurrence Equations:

A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_K a_{n-K} + F(n)$$

where $c_1, c_2, \ldots, c_K$ are real nos. & $F(n)$ is a fn. not identically zero depending only on n, is called a non homogeneous recurrence relation with constant coefficients.

1. Solve the recurrence relation $S(K) - 7S(K-1) + 10S(K-2) = 8K+6$ with $S(0)=1$, $S(1)=2$ $\rightarrow (1)$

Givn. $S(K) - 7S(K-1) + 10S(K-2) = 0$

CE: $m^2 - 7m + 10 = 0$

$(m-2)(m-5) = 0$

$m = 2, 5$

$$HS = A(2)^K + B(5)^K \rightarrow (2)$$

PS

$RHS = 8K+6$

Take

$$\left.\begin{aligned} S(K) &= cK + d \\ S(K-1) &= c(K-1) + d \\ S(K-2) &= c(K-2) + d \end{aligned}\right\} \rightarrow (A)$$

Subs. (A) in (1),

$$ck+d-7[c(k-1)+d]+10[c(k-2)+d]=8k+6$$

$$ck+d-7ck+7c-7d+10ck-20c+10d=8k+6$$

$$4ck-13c+4d=8k+6$$

Equating the coefficient of n and constant,

$$4c=8 \qquad -13c+4d=6$$
$$c=2 \qquad -13(2)+4d=6$$
$$4d=6+26$$
$$4d=32$$
$$d=8$$

$$\therefore PS=ck+d$$
$$=2k+8 \rightarrow (3)$$

General soln.

$$S(k)=A(2)^k+B(5)^k+2k+8 \rightarrow (4)$$

Givn. $S(0)=1$

$$S(0)=A+B+8=1$$
$$A+B=-7 \rightarrow (5)$$

and $S(1)=2$

$$S(1)=A(2)+B(5)+2+8=2$$
$$2A+5B=-8 \rightarrow (6)$$

Solving (5) and (6),

$$A+B=-7 \rightarrow (5)$$
$$2A+5B=-8 \rightarrow (6)$$
$$(5)\times 2 \Rightarrow \underset{(-)}{2A}+\underset{(-)}{2B}=\underset{(+)}{-}14$$
$$3B=6 \Rightarrow B=2$$

Subs. $B = 2$ in (5),

$$A + B = -7$$
$$A = -2 - 7 = -9$$
$$A = -9$$

Subs. A & B in (4),

$$S(k) = -9(2)^k + 2(5)^k + 2k + 8$$

3. Solve the Recurrence Relation

$a_n - a_{n-1} - 6a_{n-2} = -30$, $a_0 = 0$, $a_1 = -5$, $n \geq 2$

Givn. $a_n - a_{n-1} - 6a_{n-2} = -30 \rightarrow (1)$

CE: $m^2 - m - 6 = 0$

$$(m-3)(m+2) = 0$$
$$m = 3, -2$$

$$HS = A(3)^n + B(-2)^n \rightarrow (2)$$

PS

RHS = a constant

Take $a_n = a_{n-1} = a_{n-2} = d$

$(1) \Rightarrow d - d - 6d = -30$

$$-6d = -30$$
$$d = 5$$

$$PS = 5 \rightarrow (3)$$

General soln.

$$a_n = A(3)^n + B(-2)^n + 5 \rightarrow (4)$$

Givn. $a_0 = 0$

$$A + B + 5 = 0$$
$$A + B = -5 \rightarrow (5)$$

and $a_1 = -5$

$$3A - 2B + 5 = -5$$

$$3A - 2B = -10 \rightarrow (6)$$

Solving (5) and (6),

$$(5) \times 2 \Rightarrow \quad 2A + 2B = -10$$
$$3A - 2B = -10$$
$$5A = -20$$
$$A = -\frac{20}{5} = -4$$

$$(5) \Rightarrow \quad -4 + B = -5$$
$$B = -5 + 4$$
$$B = -1$$

$$(4) \Rightarrow \quad a_n = -4(3)^n - 1(-2)^n + 5$$

3]. Solve $a_n - 2a_{n-1} - 3a_{n-2} = 4^n + 6$

Gvn. $a_n - 2a_{n-1} - 3a_{n-2} = 4^n + 6 \rightarrow (1)$

CE: $m^2 - 2m - 3 = 0$

$$(m-3)(m+1) = 0$$
$$m = 3, -1$$

$$HS = A(3)^n + B(-1)^n \rightarrow (2)$$

PS:

$$RHS = 4^n + 6$$
$$PS = PS_1 + PS_2$$

PS_1: Take
$$\left.\begin{aligned} a_n &= d.4^n \\ a_{n-1} &= d.4^{n-1} \\ a_{n-2} &= d.4^{n-2} \end{aligned}\right\} \rightarrow (3)$$

Subs. (3) in (1),

$$d.4^n - 2d.4^{n-1} - 3d.4^{n-2} = 4^n$$
$$d.4^n - 2d.\frac{4^n}{4} - 3d.\frac{4^n}{16} = 4^n$$

$$4^n\left[d-\frac{d}{2}-\frac{3d}{16}\right]=4^n$$

$$\frac{16d-8d-3d}{16}=1$$

$$\frac{5d}{16}=1$$

$$d=\frac{16}{5}$$

$$PS_1=\frac{16}{5}(4)^n$$

PS_2:

RHS = a constant

Take $a_n=a_{n-1}=a_{n-2}=d$

$$d-2d-3d=6$$

$$-4d=6$$

$$d=\frac{6}{-4}$$

$$d=\frac{-3}{2}$$

$$PS_2=\frac{-3}{2} \qquad PS=\frac{16}{5}(4)^n-\frac{3}{2}$$

General soln.

$$a_n=A(3)^n+B(-1)^n+\frac{16}{5}(4)^n-\frac{3}{2}$$

4]. Solve $a_n-4a_{n-1}+4a_{n-2}=2^n+3n,\ n\geq 2$

Givn. $a_n-4a_{n-1}+4a_{n-2}=2^n+3n \to (1)$

CE: $m^2-4m+4=0$

$$(m-2)^2=0$$

$$m=2,2$$

$$HS=(A+nB)2^n$$

PS:

RHS $= 2^n + 3n$

$PS = PS_1 + PS_2$

$PS_1 = 2^n$

Take $a_n = dn^2 2^n$ since base of RHS is 2, which is a double root of CE

$a_{n-1} = d(n-1)^2 2^{n-1}$

$a_{n-2} = d(n-2)^2 2^{n-2}$

$\therefore (1) \Rightarrow dn^2 2^n - 4[d(n-1)^2 2^{n-1}] + 4[d(n-2)^2 2^{n-2}] = 2^n$

$\div 2^n \quad dn^2 - 4d(n-1)^2 2^{-1} + 4d(n-2)^2 2^{-2} = 1$

$dn^2 - \frac{4d}{2}(n^2+1-2n) + \frac{4d}{4}(n^2+4-4n) = 1$

$dn^2 - 2d(n^2+1-2n) + d(n^2+4-4n) = 1$

$dn^2 - 2dn^2 - 2d + 4dn + dn^2 + 4d - 4dn = 1$

$2d = 1$

$d = 1/2$

$PS_1 = \frac{1}{2} n^2 (2)^n$

$PS_2 = 3n$

Take $a_n = d_0 + d_1 n$

$a_{n-1} = d_0 + d_1(n-1)$

$a_{n-2} = d_0 + d_1(n-2)$

$(1) \Rightarrow d_0 + d_1 n - 4[d_0 + d_1(n-1)] + 4[d_0 + d_1(n-2)] = 3n$

$d_0 + d_1 n - 4d_0 - 4d_1 n + 4d_1 + 4d_0 + 4d_1 n - 8d_1 = 3n$

$d_0 - 4d_1 + d_1 n = 3n$

Equating the coeffi. of n and constant,

$$d_1 = 3; \quad d_0 - 4d_1 = 0$$
$$d_0 = 4d_1 = 12$$
$$d_0 = 12$$

$$PS_2 = 12 + 3n \qquad PS = \frac{1}{2}(n)^2(2)^n + 12 + 3n$$

General soln.

$$a_n = (A + nB)(2)^n + \frac{1}{2}(n)^2(2)^n + 12 + 3n$$

HW I. Solve $S(K) - 5S(K-1) + 6S(K-2) = 2$ with $S(0) = 1$, $S(1) = -1$.