Fourier series expansion in $(-l, l)$

1) Obtain the Fourier series expansion for $f(x) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l. \end{cases}$

Soln: $f(x) = \begin{cases} l+x & , -l \le x \le 0 \\ l-x & , 0 \le x \le l \end{cases}$

Here $\phi_1(x) = l+x$; $\phi_2(x) = l-x$.

$\phi_1(-x) = l-x$.
$= \phi_2(x)$, even function .

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x$.

Now $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$= \frac{2}{l}\int_0^l (l-x)\,dx$

$= \frac{2}{l}\left[lx - \frac{x^2}{2}\right]_0^l$

$= \frac{2}{l}\left[\frac{l^2}{2}\right] = l$

$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$

$= \frac{2}{l}\int_0^l (l-x)\cos\left(\frac{n\pi}{l}\right)x\,dx$

$= \frac{2}{l}\left[(l-x)\left(\frac{\sin(n\pi/l)x}{\left(\frac{n\pi}{l}\right)}\right) - (-1)\left(-\frac{\cos(n\pi/l)x}{(n\pi/l)^2}\right)\right]_0^l$

$$= \frac{2}{l}\left[-\frac{l^2}{(n\pi)^2}(\cos n\pi - \cos 0)\right]$$

$$= \frac{-2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \frac{2l}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$\therefore f(x) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi}{l}\right)x.$$

2) Obtain the Fourier series expansion for $f(x) = x^2$ in $(-l, l)$

Soln: $a_0 = \frac{2}{3} l^2$.

$a_n = \frac{4l^2}{n^2\pi^2}(-1)^n$

3) Obtain the Fourier series for $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 1, & 0 < x < 2. \end{cases}$

Soln: Here $l = 2$. (neither odd nor even)

$a_0 = 1$

$a_n = 0$

$b_n = -\frac{1}{n\pi}\left[(-1)^n - 1\right]$