UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
ONE DIMENSIONAL EQUATION OF HEAT CONDUCTION

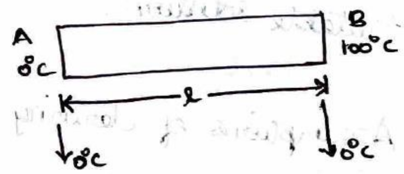
Steady state condition:-
The state in which the temperature depends only on the distance but not on time t, is called steady state.
Therefore $u(x,t)$ becomes $u(x)$ under the steady state.

Note: $u(x) = \left(\frac{b-a}{l}\right)x + a.$

Type 1: [Steady state conditions bothends at zero temperature]

1. A rod of length l has its ends A and B kept at $0°c$ and $100°c$ until steady state condition prevail. If the temperature at B is reduced suddenly to $0°c$ and kept so, while that of A is maintained, find the temperature $u(x,t)$ at a distance x from A and at time t.

Solution :-



The ODHE is
$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are

i) $u(0,t)=0$

ii) $u(l,t)=0$

iii) $u(x,0) = \frac{100x}{l}$

$$f(x) = \left(\frac{b-a}{l}\right)x + a$$
$$= \left(\frac{100-0}{l}\right)x + 0$$
$$= \frac{100x}{l}$$

The suitable solution is
$$u(x,t) = (A\cos px + B\sin px)\, e^{-a^2p^2t} \longrightarrow (1)$$

Applying i) in (1) we get $u(0,t)=0$
$$(A+0)\, e^{-a^2p^2t} = 0.$$

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Here $e^{-a^2\lambda^2 t} \neq 0$ [since it is a fn of time]

and $B \neq 0$. [Suppose $B=0$, we get trivial solution]

$\therefore \sin \lambda 30 = 0$

$\lambda(30) = \sin^{-1} 0$

$\lambda(30) = n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{30}}$

Sub $\lambda = \frac{n\pi}{30}$ in ②

$$u(x,t) = B \sin \frac{n\pi x}{30} e^{-a^2 \frac{n^2\pi^2}{900} t}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} e^{-a^2 \frac{n^2\pi^2}{900} t}$$

Apply (iii) in ③

$u(x,0) = f(x)$ in ③

$= 2x + 20$

$$u(x,0) \Rightarrow \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} e^{0}$$

$$= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30}$$

$$\therefore \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} = 2x + 20$$

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{30} = 2x + 20 \qquad B_n = b_n$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$

$$\therefore b_n = \frac{2}{30} \int_0^{30} (2x+20) \sin \frac{n\pi x}{30} dx$$

$$= \frac{1}{15} \left[(2x+20) \left(\frac{-\cos \frac{n\pi x}{30}}{n\pi/30} \right) - (2) \left(\frac{-\sin \frac{n\pi x}{30}}{(n\pi/30)^2} \right) \right]_0^{30}$$

Half Range sine series

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \frac{100x}{l} \qquad \therefore B_n = b_n$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$B_n = \frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{200}{l^2}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{200}{l^2}\left[x\frac{-\cos\frac{n\pi x}{l}}{n\pi/l} - 1\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right]_0^l$$

$$= \frac{200}{l^2}\left[\frac{-lx}{n\pi}\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{200}{l^2}\left[\frac{-l^2}{n\pi}(-1)^n\right]$$

$$= \frac{-200}{n\pi}(-1)^n$$

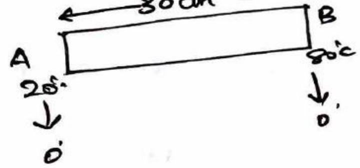
$$B_n = \frac{200(-1)^{n+1}}{n\pi}$$

Sub B_n value in (4), we get

$$u(x,t) = \sum_{n=1}^{\infty}\frac{200(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{l}\, e^{-\frac{a^2n^2\pi^2 t}{l^2}}$$

$$= \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\, e^{-\frac{a^2n^2\pi^2 t}{l^2}}$$

2. A rod 30cm long has its ends A and B kept at 20° and 80° respectively. Until steady state conditions prevails. The temperature at each end is then suddenly reduced to 0°c and kept so. Find the resulting temperature function $u(x,t)$ taking $x=0$ at A.



Solution:

The ODHE is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are

i) $u(0,t)=0,\ t>0$

(ii) $u(30,t)=0,\ t>0$

(iii) $u(x,0) = 2x+20$

$$f(x) = \left(\frac{b-a}{l}\right)x + a$$

$$= \left(\frac{80-20}{30}\right)x + 20$$

$$= \left(\frac{60}{30}\right)x + 20$$

$$= 2x+20$$

The suitable soln is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-a^2\lambda^2 t} \rightarrow ①.$$

Applying (i) in ①

$u(0,t)=0.$

$$u(0,t) \Rightarrow (A(1)+B(0))\, e^{-a^2\lambda^2 t} = 0.$$

$$A\, e^{-a^2\lambda^2 t} = 0.$$

$e^{-a^2\lambda^2 t} \neq 0$ [since it is a fn of time]

$$\boxed{A=0}$$

Sub $A=0$ in ①

$$u(x,t) = B\sin\lambda x\, e^{-a^2\lambda^2 t} \rightarrow ②.$$

Apply (ii) in ②

$u(30,t)=0.$

$$u(30,t) = B\sin\lambda 30\, e^{-a^2\lambda^2 t} = 0$$

$$A\,e^{-a^2p^2t} = 0$$

Here $e^{-a^2p^2t} \neq 0$ [It is a function of time]

$$\Rightarrow A = 0$$

Sub $A = 0$ in ① we get

$$u(x,t) = B\sin px\, e^{-a^2p^2t} \rightarrow ②$$

Applying (ii) in ② we get

$$u(l,t) = 0$$

$$B\sin pl\, e^{-a^2p^2t} = 0$$

Here $e^{-a^2p^2t} \neq 0$ [It is a function of t]

$B \neq 0$ [Suppose $B = 0$, we get a trivial soln]

$$\Rightarrow \sin pl = 0$$

$$pl = \sin^{-1} 0$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Sub $p = \frac{n\pi}{l}$ in ②

$$u(x,t) = B\sin\frac{n\pi x}{l}\, e^{-\frac{a^2n^2\pi^2}{l^2}t} \rightarrow ③$$

The most general soln is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{-\frac{a^2n^2\pi^2}{l^2}t} \rightarrow ④$$

Applying iii) in ④

$$u(x,0) = \frac{100x}{l}$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{100x}{l}$$

$$= \frac{1}{15}\left[\left\{[2(30)+20]\left[\frac{-\cos\frac{n\pi 30}{30}}{\frac{n\pi}{30}}\right] + 2\left[\frac{\sin\frac{n\pi 30}{30}}{n^2\pi^2/900}\right]\right\} - \left\{30\left(\frac{-\cos 0}{n\pi/30}\right) + 2\left(\frac{\sin 0}{n^2\pi^2/900}\right)\right\}\right]$$

$$= \frac{1}{15}\left[-80(-1)^n\left(\frac{30}{n\pi}\right) + 20(1)\frac{30}{n\pi}\right]$$

$$= \frac{1}{15}\left[-\frac{2400}{n\pi}(-1)^n + \frac{600}{n\pi}\right]$$

$$= \frac{1}{15}\left[\frac{600}{n\pi}\right]\left[1-4(-1)^n\right]$$

$$= \frac{40}{n\pi}\left[1-4(-1)^n\right] \rightarrow (4)$$

$$(4) \Rightarrow u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left[1-4(-1)^n\right] \sin\frac{n\pi x}{30}\, e^{-a^2\frac{n^2\pi^2}{900}t}$$