SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

COIMBATORE – 35

UNIT – II

FOURIER SERIES

TWO MARKS

1. Explain periodic function with two examples.

Solution: A function $f(x)$ is said to have a period T if for all $x, f(x+T) = f(x)$,

Where T is a positive constant. The least value of $T > 0$ *is called the period of* $f(x)$.

For examples, $f(x) = \text{Sin}x$

$f(x+2\pi) = \text{Sin}(x+2\pi) = \text{Sin}x$

Here, $f(x) = f(x+2\pi)$

2. State Dirichlet's condition for a given function to expend in Fourier series.

Solution: Any function f(x) can be developed as a Fourier series, provided

i) f(x) is periodic, single valued & finite.
ii) f(x) has a finite number of discontinuities in any one period
iii) f(x) has a finite number of maxima and minima

3. State general Fourier series.

solution: The Fourier series of $f(x)$ in $c \le x \le c + 2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Where a_0, an &b_n are called Fourier coefficients(or) Euler constants

4. Find the coefficient of b_n of $\cos 5x$ in the Fourier cosine series of the function

$f(x) = \sin 5x$ in the in the interval $(0, \pi)$.

Solution: The Fourier Cosine series is

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} \cos 5x \cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} [\cos(5+n)x + \cos(5-n)x] d$$

$$= \frac{2}{\pi}\left[\frac{\sin(5+n)x}{5+n} + \frac{\sin(5-n)x}{5-n}\right]_0^{\pi} = 0; \text{Therefore}, b_n = 0$$

5. Find the constant a_0 of the Fourier series for the function of $f(x) = x$ in $0 \le x \le 2\pi$

Solution:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} xdx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi} = 2\pi$$

6. Obtain the first term of the Fourier series for the function $f(x) = x^2, -\pi < x < \pi$.

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2}{3}\pi^2$$

7. If $f(x) = x^3, -\pi < x < \pi$. *Find the constant term of its Fourier series.*

Solution:

$f(x) = x^3$, is an odd function. Therefore, the fourier constants $a_0 = 0$

8. Find the constant term in the Fourier series expansion of $f(x) = x$ in $(-\pi, \pi)$

Solution:

$a_0 = 0$ since $f(x)$ is an odd function $(-\pi, \pi)$

9. Find the sum of the Fourier series of $f(x) = x + x^2$ in $-\pi < x < \pi$ *at* $x = \pi$.

Solution:

Given $f(x) = x + x^2$

The sum of Fourier series is equal to the arithmetic mean of the value of $f(x)$ at $x = \pi$ and $x = -\pi$.

Sum of Fourier series $= \frac{f(\pi)+f(-\pi)}{2} = \frac{\pi+\pi^2-\pi+\pi^2}{2} = \pi^2$

10. What is the constant term a_0 and the coefficient of cosnx, a_n in the Fourier series of $f(x) = x - x^3$ in $(-\pi,\pi)$.

Solution:

$$f(x) = x - x^3 \Rightarrow f(-x) = -x + x^3$$

$$= -(x - x^3) = -f(x)$$

Therefore, f(x) is an odd function of x in $(-\pi,\pi)$. Therefore, the Fourier series of f(x) Contains sine terms only. Therefore, $a_0 = 0$ and $a_n = 0$

11. Find b_n in the expansion of x^2 as a Fourier series in $(-\pi,\pi)$.

Solution:

$f(x) = x^2 \Rightarrow f(-x) = x^2 = f(x)$

Therefore, f(x) is an even function of x in $(-\pi,\pi)$. The coefficient b_n of Sinnx in the Fourier expansion is zero. Therefore, $b_n = 0$

12. Find a_n in expanding e^{-x} as Fourier series in $(-\pi,\pi)$.

Solution:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\text{Cos } nx + n \text{ Sin } nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[-e^{-\pi}(-1)^n + (-1)^n e^{\pi}\right]$$

$$a_n = \frac{(-1)^n (e^{\pi} - e^{-\pi})}{\pi(1+n^2)} = \frac{2(-1)^n \text{Sinh } \pi}{\pi(1+n^2)}$$

13. Fine the Fourier constant b_n for x Sin x in $(-\pi,\pi)$.

Solution:

Let $f(x) = x \operatorname{Sin} x$, Therfore, $f(x) = (-x)\operatorname{Sin}(-x) = x \operatorname{Sin} x = f(x)$

Therefore, $f(x)$ is even function of x in $(-\pi, \pi)$.

Therefore, $b_n = 0$

14. If $f(x) = |x|$ is expanded as a Fourier series in $(-\pi, \pi)$, find the value of a_n?

Solution:

$f(x) = |x|$ is an odd function in $(-\pi, \pi)$.

Therefore, the value of the Fourier coefficient $a_n = 0$.

15. Suppose the function x cos x has the series expansion $\sum_{n=1}^{\infty} b_n$ Sin x in $(-\pi, \pi)$, find the value of b_1.

Solution: $b_1 = \frac{2}{\pi}\int_0^{\pi} x\operatorname{Cos}x\operatorname{Sin}x\,dx = \frac{1}{\pi}\int_0^{\pi} x\operatorname{Sin}2x\,dx = \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) + \left(\frac{\operatorname{Sin}2x}{4}\right)\right]_0^{\pi}$

$$= \frac{1}{\pi}\left(\frac{-\pi}{2}\right) = \frac{-1}{2}$$

16. Find the value of a_n in the Fourier expansion of f(x)= x^2 in $(0, 2\pi)$

Solution:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \operatorname{Cos}nx\,dx = \frac{1}{\pi}\left[x^2\left(\frac{\operatorname{Sin}nx}{n}\right) - (2x)\left(\frac{-\operatorname{Cos}nx}{n^2}\right) + 2\left(\frac{\operatorname{Sin}nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right) = \frac{4}{n^2}$$

17. Does f(x) = tan x possess a Fourier expansion in (0, π).

Solution:

f(x) =tan x has an infinite discontinuity at $x = \frac{\pi}{2}$

Since, the Dirichlet`s conditions on continuity is not satisfied, the function f(x)=tan x has no Fourier expansion.

18. Find a Sine series for the function f(x)=1,0 < x < π.

Solution:

$$b_n = \frac{2}{\pi}\int_0^{\pi} \text{Sinnxdx} = \frac{2}{\pi}\left[-\frac{\text{Cosnx}}{n}\right]_0^{\pi} = \frac{2}{n\pi}[1-(-1)^n]$$

$$= \frac{4}{n\pi} \text{ when n is odd}$$

Therefore, the Fourier sine series of $f(x) = \sum_{n=1}^{\infty} b_n \text{Sinnx}$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi}\frac{\text{Sinnx}}{n} = \frac{4}{\pi}\left[\text{Sinx} + \frac{\text{Sin3x}}{3} + \frac{\text{Sin5x}}{5} + \cdots\right]$$

19. When an even function f(x) is expanded in a Fourier series in the interval from –π to π. Show that b_n=0

Solution:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \text{Sinnx dx}$$

Since f(x) is even and Sin nx is odd then product f(x) Sin nx is an odd function.

By Property of definite integral b_n =0.

20. If f(x) is an odd function defined in (-1,1) What are the values of a_0 and a_n

Solution:

$a_0 = 0$ and $a_n = 0$ since f(x) is an odd function.

21.Find the root mean square value of the function f(x)= x^2 in the interval (0,1).

Solution:

$$\text{RMS value} = \sqrt{\frac{1}{l}\int_0^1 x^2\,dx} = \sqrt{\frac{1}{l}\left(\frac{x^3}{3}\right)_0^1} = \sqrt{\frac{1}{l}\left[\frac{l^3}{3}\right]} = \frac{1}{\sqrt{3}}$$

22. Define root mean square value of a function f(x) in a < x < b.

Solution:

$$\text{R.M.S.value } \bar{y} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}$$

23 .State Parseval`s identity for full range expansion of f(x) as Fourier series in (0,2l).

Solution:

$$\frac{1}{l}\int_0^{2l}(f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \text{ where } a_0, a_n \text{ and } b_n \text{ are}$$

Fourier coefficients in the expansion of $f(x)$ as a Fourier series .

24 . State Parseval`s identity of Fourier series.

Solution:

If f(x) has a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty}(a_n \text{Cos}nx + b_n \text{Sin}nx) \text{ in } (0,2\pi), \text{then}$$

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

25. What do you mean by Harmonic Analysis?

Solution:

The process of finding the Fourier series for a function given by numerical value is known as harmonic analysis. In harmonic analysis

$$a_0 = 2\,(\text{mean value of y in } (0,2\pi))$$

$a_n = 2$ (mean value of y Cos nx in $(0,2\pi)$)

$b_n = 2$(mean value of y Sin nx in $(0,2\pi)$)

PART C
MODEL QUESTIONS

1. Find the Fourier series for $f(x) = |\cos x|$ in the interval $(-\pi, \pi)$.

2. Find the Fourier series for $f(x) = \begin{cases} l - x & \text{in } 0 \le x \le l \\ 0 & \text{in } l \le x \le 2l \end{cases}$. Hence deduce the sum to infinity of the series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

3. Obtain the Fourier series for $f(x)$ of period 2l and defined as follows

$$f(x) = \begin{cases} L + x & \text{in } (-L, 0) \\ L - x & \text{in } (0, L) \end{cases} \quad \text{Hence deduce that } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

4. Show that for 0 < x < l, $x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \cdots\right)$. Deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$.

5. Find the Fourier series for the function $f(x) = \begin{cases} x & \text{in } 0 < x < 1 \\ 1\text{-}x & \text{in } 1 < x < 2 \end{cases}$

6. Find the Fourier series of $f(x) = x^2$ in $-\pi < x < \pi$ Hence find ,

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots\ldots\ldots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$

7. Find the Fourier series for the function

$$f(x) = \begin{cases} 1+\frac{2x}{\pi} & \text{in } -\pi < x < 0 \\ 1-\frac{2x}{\pi} & \text{in } 0<x < \pi \end{cases}$$

8. Find the Fourier series for the function f(x) =x sin x , $0 < x < 2\pi$ and hence show that
$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots = \frac{\pi-2}{4}$

9. Find the Fourier series for the function $f(x) = x(\pi^2 - x^2)$ in $(-\pi, \pi)$.

10.Find the sine series of f(x) = $\begin{cases} x-1, & 0 \le x \le 1 \\ 1-x & 1 \le x \le 2 \end{cases}$

11. Find the Fourier series expansion of period 2L for the function

f(x) = $(L - x)^2$ in the range (0, 2L). Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ **.**

12. Find the Fourier series of f(x) = $\begin{cases} -K & \text{in } (-\pi, 0) \\ K & \text{in } (0, \pi) \end{cases}$

13. Obtain Fourier series for f(x) of period 2L and defined as follows:

f(x) = $\begin{cases} L - x & \text{in } (0, L) \\ 0 & \text{in } (L, 2L) \end{cases}$ **. Hence deduce that** $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$**.**

14. Find a cosine series for the function f(x) = x^2 in (0, π)

15. Determine the Fourier series expansion of f(x)= $\left(\frac{\pi - x}{2}\right)^2$ **in** $0 < x < 2\pi$ **with period** 2π

16. Find the Fourier series as far as the second harmonic to represent the function given in the following data.

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

17. Find the Fourier series expansion defined in $(0, T)$ by means of the table of values given below. Find the series up to the first harmonic.

t-Sec	0	T/6	T/3	T/2	2T/3	5T/6	T
A - amp	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

18. Find the Fourier Series up to second Harmonic level for the following data:

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
f(x)	1.0	1.4	1.9	1.7	1.5	1.2	1.0