UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
SOLUTIONS OF ONE DIMENSIONAL WAVE EQUATION

One Dimensional Wave equation [Hyperbolic]

Let $y(x,t)$ represents the 1-D wave equation

(i.e) $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$

where $a^2 = \dfrac{T}{M} = \dfrac{\text{Tension}}{\text{Mass per unit length}}$.

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Possible solution of 1-D wave equation:-

1. $y(x,t) = (A_1 e^{px} + A_2 e^{-px})(A_3 e^{pat} + A_4 e^{-pat})$

2. $y(x,t) = A_5(\cos px + A_6 \sin px)(A_7 \cos pat + A_8 \sin pat)$

3. $y(x,t) = (A_9 x + A_{10})(A_{11} t + A_{12})$

Suitable Solution:-

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$

Assumptions for deriving 1-Dimensional wave equation:

1. The motion takes place entirely in one plane.
i.e) xy-plane.
2. The tension 'T' is constant at all times and at all points of the deflected string
3. The effect of friction is negligible.
4. The string is perfectly flexible.
5. The slope of the deflection curve at all points is neglectable.

Boundary conditions

Given Displacement

i) $y(0,t) = 0,\ t > 0$

(ii) $y(l,t) = 0,\ t > 0$

iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0},\ 0 < x < l$

iv) $y(x,0) = f(x)$

Given Velocity

i) $y(0,t) = 0,\ t > 0$

ii) $y(l,t) = 0,\ t > 0$

iii) $y(x,0) = 0$

iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = f(x),\ 0 < x < l.$

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1. A string is stretched & fastened to two points $x=0$ & $x=l$ apart motion is started by displacing the string into the form $y = K(lx - x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of x from one end at time t.

One dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \text{ where } a^2 = \frac{T}{M}$$

$\therefore$ The suitable solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \rightarrow (1)$$

Boundary condition:

i) $y(0,t) = 0, \ t > 0$

ii) $y(l,t) = 0, \ t > 0$

Initial condition:-

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \ 0 < x < l$

$\frac{\partial}{\partial t} y(x,0) = 0.$

iv) $y(x,0) = f(x), \ 0 < x < l$

$= K(lx - x^2), \ 0 < x < l.$

Applying condition i) in eqn (1)

$$y(0,t) = (A\cos\lambda(0) + B\sin\lambda(0))(C\cos\lambda at + D\sin\lambda at)$$

$$0 = A(C\cos\lambda at + D\sin\lambda at)$$

$A = 0$ [since Boundary condition]

$C\cos\lambda at + D\sin\lambda at \neq 0$

Applying $A=0$ to eqn (1)

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \rightarrow (2)$$

Applying condition ii) in eqn ②

$$y(l,t) = B\sin\lambda l\,(c\cos\lambda at + D\sin\lambda at)$$

$$0 = B\sin\lambda l\,(c\cos\lambda at + D\sin\lambda at)$$

$B \neq 0$, $\sin\lambda l = 0$, $c\cos\lambda at + D\sin\lambda at \neq 0.$

↳ only one constant should be zero (0)

$$\sin\lambda l = 0$$

$$\lambda l = \sin^{-1}(0)$$

$$\lambda l = n\pi \qquad \boxed{\lambda = \frac{n\pi}{l}}$$

Applying λ value in eqn ②

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(c\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \to ③$$

Diff pwrt 't',

$$\frac{\partial y}{\partial t}(x,t) = B\sin\frac{n\pi x}{l}\left(-c\sin\frac{n\pi at}{l}\cdot\frac{n\pi a}{l} + D\cos\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right)$$

when $t=0$,

$$\frac{\partial y}{\partial t}(x,0) = B\sin\frac{n\pi x}{l}\left(-c\sin\frac{n\pi a(0)}{l}\cdot\frac{n\pi a}{l} + D\cos\frac{n\pi a(0)}{l}\cdot\frac{n\pi a}{l}\right)$$

$$0 = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$B\sin\frac{n\pi x}{l} \neq 0$, $D = 0$, $\frac{n\pi a}{l} \neq 0.$

$$y(x,t) = B\sin\frac{n\pi x}{l}\cdot c\cos\frac{n\pi at}{l}$$

$$= Bc\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi at}{l}$$

$$\boxed{Bc = b_n}$$

$$y(x,t) = b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \quad \rightarrow (4)$$

Applying condition (iv) to equation (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a(0)}{l}$$

$$K(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Expand b_n in Half range series,

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_0^l K(lx - x^2) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2K}{l}\int_0^l (lx - x^2) \sin\frac{n\pi x}{l}\, dx$$

By applying Bernoulli's formula,

$u = lx - x^2 \qquad v = \sin\frac{n\pi x}{l}$

$u' = l - 2x \qquad v_1 = \dfrac{-\cos\frac{n\pi x}{l}}{n\pi/l}$

$u'' = -2$

$v_2 = \dfrac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}, \quad v_3 = \dfrac{\cos\frac{n\pi x}{l}}{(n\pi/l)^3}$

$$b_n = \frac{2K}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{(n\pi/l)^3}\right)\right]_0^l$$

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$$= \frac{2K}{l}\left[\left(0+0-2\frac{\cos n\pi}{(n\pi/l)^3}\right)-\left(0+0-\frac{2\cos 0}{(n\pi/l)^3}\right)\right]$$

$$= \frac{2K}{l}\left[\frac{-2(-1)^n}{(n\pi/l)^3}+\frac{2}{(n\pi/l)^3}\right] = \frac{4K}{l}\left(\frac{-l^3}{n^3\pi^3}\right)\left[(-1)^n+1\right]$$

$$= \frac{4K}{l}\times\frac{l^3}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \frac{4Kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} \frac{8Kl^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

applying b_n in equation (4)

$$y(x,t) = \sum_{n=odd}\frac{8Kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l} \rightarrow (5).$$

2. A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially in a position given by $y(x,0) = y_0\sin^3\frac{\pi x}{l}$. If it is released from rest from this position, then find the displacement.

The one dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

i) $y(0,t)=0$, $\forall t$ ii) $y(l,t)=0$, $\forall t$

iii) $\frac{\partial}{\partial t}y(x,0)=0$, $\forall x$

iv) $y(x,0) = y_0\sin^3\frac{\pi x}{l}$.

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The suitable solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \rightarrow ①$$

Applying (i) in ① we get

$$y(0,t) = 0$$

$$(A(1) + B(0))(C\cos pat + D\sin pat) = 0$$

$$A(C\cos pat + D\sin pat) = 0$$

$C\cos pat + D\sin pat \neq 0$ (It is a function of time)

$$\Rightarrow A = 0.$$

Sub $A = 0$ in ①

$$y(x,t) = B\sin px\,(C\cos pat + D\sin pat) \rightarrow ②$$

Applying (ii) in ② $y(l,t) = 0$

$$B\sin pl\,(C\cos pat + D\sin pat) = 0$$

$B \neq 0$ (If $B = 0$, we get a trivial solution)

$C\cos pat + D\sin pat \neq 0$ ($\because$ It is a function of 't')

$$\Rightarrow \sin pl = 0.$$

$$pl = \sin^{-1}(0) \Rightarrow pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Sub $p = \frac{n\pi}{l}$ in ②

$$y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right] \rightarrow ③$$

Before applying (iii) diff ③ p wrt 't'

$$\frac{\partial}{\partial t} y(x,t) = B\sin\frac{n\pi x}{l}\left[-C\sin\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right) + D\cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)\right]$$

Applying (iii) we get,

$$\frac{\partial}{\partial t} y(x,0) = 0$$

$$B \sin \frac{n\pi x}{l} \left[0 + D \frac{n\pi a}{l} \right] = 0.$$

$$BD \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = 0.$$

Here $B \neq 0$ [If $B=0$, we get a trivial soln]

$\sin \frac{n\pi x}{l} \neq 0$ [$\because$ It is a function of x]

$\Rightarrow D = 0$

Sub $D=0$ in (3)

$$y(x,t) = B \sin \frac{n\pi x}{l} \; C \cos \frac{n\pi at}{l}$$

$$= BC \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \rightarrow (4)$$

Applying condition (iv) in (4) we get

$$y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \frac{y_0}{4} \left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right]$$

$$\Rightarrow B_1 \sin \frac{\pi x}{l} + B_2 \sin \frac{2\pi x}{l} + B_3 \sin \frac{3\pi x}{l} + \dots$$

$$= \frac{3y_0}{4} \sin \frac{\pi x}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l}$$

Equating the like coeff we get,

$$B_1 = \frac{3y_0}{4} \; ; \; B_2 = 0 \; ; \; B_3 = \frac{-y_0}{4} \; ; \; B_4 = B_5 = \dots = 0.$$

Sub the above values in (4)

$$y(x,t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l}$$