## SOLUTION OF TWO DIMENSIONAL HEAT FLOW EQUATION

The two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The possible solutions of two dimensional heat equation is

(i) $u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$

(ii) $u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$

(iii) $u(x,y) = (Ax + B)(Cy + D)$

The suitable soln. is TYPE-I Heat flows in x direction (ie) $0<x<l$

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}).$$

The boundary cdtns: are:

i) $u(0,y) = 0$

ii) $u(l,y) = 0$

iii) $u(x,0) = 0$

iv) $u(x,l) = f(x)$ $0<x<l$.

A square plate is bdd. by the lines $x=0$, $y=0$, $x=20$ and $y=20$. Its faces are insulated. The temp. along the upper horizontal edge is gn. by $u(x,20) = x(20-x)$ when $0<x<20$ while the other three edges are kept at $0^\circ C$. Find the steady state temp. in the plate.

Soln: Let $u(x,y)$ be the temp. at any point $(x,y)$.

Then $u(x,y)$ satisfies the Laplace's eqn.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0$$

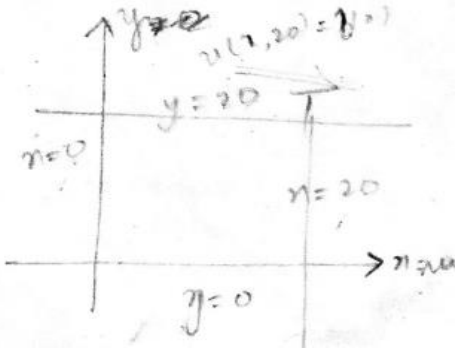
The boundary coditns. are:

(i) $u(0,y) = 0$

(ii) $u(20,y) = 0$

(iii) $u(x,0) = 0$

(iv) $u(x,20) = x(20-x)$, $0 < x < 20$




The suitable soln. is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \quad \text{——} (1)$$

Apply (i) in (1)

$$u(0,y) = A(Ce^{py} + De^{-py})$$

$$0 = A(Ce^{py} + De^{-py}) \Rightarrow \boxed{A = 0}$$

$$\therefore u(x,y) = B\sin px\,(Ce^{py} + De^{-py}) \quad \text{——} (2)$$

Apply (ii) in (2)

$$u(20,y) = B \sin 20p \left(Ce^{py} + De^{-py}\right)$$

$$0 = B \sin 20p \left(Ce^{py} + De^{-py}\right)$$

$\Rightarrow B \neq 0, \sin 20p = 0$

$$\sin 20p = \sin n\pi$$

$$\boxed{p = \frac{n\pi}{20}}$$

$$\therefore u(x,y) = B \sin \frac{n\pi}{20} x \left(Ce^{\frac{n\pi}{20}y} + De^{-\frac{n\pi}{20}y}\right) \quad \text{---(3)}$$

Apply (iii) in (3)

$$u(x,0) = B \sin \frac{n\pi}{20} x \left(C + D\right)$$

$$0 = B \sin \frac{n\pi}{20} x \left(C + D\right)$$

$\Rightarrow C + D = 0$

$\Rightarrow \boxed{D = -C}$

$$\therefore u(x,y) = B \sin \frac{n\pi}{20} x \left(Ce^{\frac{n\pi}{20}y} - Ce^{-\frac{n\pi}{20}y}\right)$$

$$= BC \sin \frac{n\pi}{20} x \left(e^{\frac{n\pi}{20}y} - e^{-\frac{n\pi}{20}y}\right)$$

$$= BC \sin \frac{n\pi}{20} x \left(2 \cdot \sin h \frac{n\pi y}{20}\right)$$

$$u(x,y) = 2BC \sin \frac{n\pi}{20} x \sin h \frac{n\pi y}{20}$$

∴ The general soln. is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{20}x \sin\frac{hn\pi}{20}y \quad \text{——— (4)}$$

Apply (iv) in (4)

$$u(x,20) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{20}x \sin\frac{hn\pi}{20}.20$$

$$= \sum_{n=1}^{\infty} A_n \sin hn\pi \sin\frac{n\pi x}{20}$$

$$x(20-x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{20} \quad \text{where } B_n = A_n \sin hn\pi .$$

$$B_n = \frac{2}{20}\int_0^{20} x(20-x)\sin\frac{n\pi x}{20}\,dx$$

$$= \frac{1}{10}\int_0^{20} (20x - x^2)\sin\frac{n\pi x}{20}\,dx .$$

$$= \frac{1}{10}\left[20x\left(-\cos\frac{n\pi}{20}x\right)\frac{20}{n\pi} - 20\left(-\sin\frac{n\pi x}{20}\right)\left(\frac{20}{n\pi}\right)^2\right]_0^{2}$$

$$- \frac{1}{10}\left[x^2\left(-\cos\frac{n\pi}{20}x\right)\cdot\frac{20}{n\pi} - 2x\left(-\sin\frac{n\pi x}{20}\right)\left(\frac{20}{n\pi}\right)^2 + 2\left(\cos\frac{n\pi}{20}x\right)\left(\frac{20}{n\pi}\right)^3\right]_0^{20}$$

$$= \frac{1}{10}\left[-400(-1)^n \frac{20}{n\pi} + 0\right] - \frac{1}{10}\left[400(-1)^n \cdot \frac{20}{n\pi} + 2(-1)^n \left(\frac{20}{n\pi}\right)^3\right]$$

$$= \frac{1}{10}\left[-400(-1)^n \frac{20}{n\pi} + 400(-1)^n \frac{20}{n\pi} - 2(-1)^n \frac{(20)^3}{(n\pi)^3} + 2\left(\frac{20}{n\pi}\right)^3\right]$$

$$= \frac{1}{5}\left[1-(-1)^n\right]\left(\frac{20}{n\pi}\right)^3$$

$$A_n = \frac{B_n}{\sinh n\pi} = \frac{1}{5}\cdot\frac{\left[1-(-1)^n\right]}{\sinh n\pi}\left(\frac{20}{n\pi}\right)^3$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{1}{5}\cdot\frac{\left[1-(-1)^n\right]}{\sinh n\pi}\cdot\frac{1600}{n^3\pi^3}\cdot \sinh\frac{n\pi y}{20}\sin\frac{n\pi x}{20}$$

$$= \sum_{n=1}^{\infty} \frac{1600}{n^3\pi^3}\cdot\frac{\left[1-(-1)^n\right]}{\sinh n\pi}\cdot \sinh\frac{n\pi y}{20}\sin\frac{n\pi x}{20}$$