DEPARTMENT OF AEROSPACE ENGINEERING

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UNIT III - DIMENSIONAL ANALYSIS AND MODEL STUDIES

DIMENSIONLESS NUMBERS (FORCE RATIOS)

When a mass is in motion, inertial force always exists. Hence, in order to develop the conditions for dynamic similarity, the ratio of inertial force and any one of the remaining forces listed previously is considered. Each of these ratios will obviously be a non-dimensional factor. The various force ratios are discussed herein:

(a) Inertia force – viscous force ratio (Reynolds number)

We know that, Inertia force = mass x acceleration

Since, mass density ρ = mass / volume, mass can be expressed as the product of mass density ρ and volume. Acceleration is the rate of change of velocity. Hence, we have,

Inertia force = (mass density x volume) (velocity / time)
= mass density x (volume / time) x velocity

By definition, (volume / time) represents the discharge. Discharge is the productof cross-sectional area of flow, *A* and the velocity of flow, *V*, i.e., discharge, $Q = AV$

So, Inertia force = $\rho (AV) V = \rho AV^2$

As cross sectional area of flow passage, A, has dimensions of L^2, we have,Inertia force, $F_i = \rho L^2V^2$

By definition, as per Newton's law of viscosity, we have, shear stress due to viscous force, F_v, is given by

$\tau = \mu (dV / dy)$

where, ρ = coefficient of viscosity of fluid (or) simply, the dynamic viscosity ofthe fluid

(dV / dy) = velocity gradient

Viscous force, F_v = shear stress x area = $\tau A = \mu (dV / dy) A$

Assuming (dV / dy) to be linear, the above expression can be written as

$F_v = \mu (V / y) A$

As y represents the thickness of fluid film, it has dimensions of L. The dimension of the area 'A' is 'L^2'. Replacing 'y' by 'L' and 'A' by 'L^2' , the above expression for F_v becomes

$F_v = \mu (V / L) L^2 = \mu VL$

Now, the ratio between the inertia force, F_i, and the viscous force, F_v, is given by

$(F_i / F_v) = \rho L^2V^2) / \mu VL) = \rho LV / \mu) = (VL / v)$

where v is the kinematic viscosity of the fluid.

The force ratio (or) non-dimensional ratio, $(\rho LV / \mu)$, is called the ***Reynoldsnumber***, Re or N_R.

The Reynolds number indicates the relative predominance of the inertia force tothe viscous force occurring in the flow system. If the Reynolds number is larger,greater will be the relative magnitude of inertia force. If the Reynolds number issmaller, the greater will be the relative magnitude of viscous force.

(b) Inertia force – Gravity force ratio (Froude number)

From the previous discussion, we have, $F_i = \rho L^2V^2$

As per Newton's second law of motion, force due to gravity can be expressed as

F_g = mass x acceleration due to gravity

Mass can be expressed as the product of mass density, ρ and the volume; hence,

F_g = (mass density x volume) (acceleration due to gravity)
= (ρ x volume) x g

Volume has dimensions of L^3. Replacing 'volume' by 'L^3', the above expressionbecomes

$F_g = \rho L^3 g$

Now, the ratio between the inertia force, F_i, and the gravity force, F_g, is given by

$(F_i / F_g) = (\rho L^2V^2) / (\rho L^3 g) = V^2 / Lg$
The square root of this ratio, i.e., $(V^2 / Lg)^{1/2} = V / (Lg)^{1/2}$ is called the ***Froude number.***

(c) Inertia Force – Pressure Force ratio (Euler number)

Pressure force, F_p can be expressed as the product of the pressure intensity, p and the area, A, over which it acts. i.e.,

$F_p = p \times A$

Area A has dimensions of L^2; Replacing 'A' by 'L^2', the above expression becomes

$F_p = p \times L^2$

Hence, the ratio between the inertia force, F_i, and the pressure force, F_p, is givenby

$F_i / F_p = (\rho L^2 V^2) / (pL^2) = \rho V^2 / p = V^2 / (p / \rho)$

The square root of this ratio, i.e., $[V^2 / (p / \rho)]^{1/2} = [V / (p/ \rho)^{1/2}]$ is called the ***Euler number***, *Eu* or N_E. The reciprocal of Euler number, i.e., $[(p / \rho)^{1/2} / V]$ is sometimes known as 'Newton number'.

(d) Inertia force – Elasticity force ratio (Mach number)

Force due to elasticity, F_e, is expressed as the product of the bulk modulus of elasticity, K, of the flowing fluid and the area, A, over which the force acts, i.e.,

$F_e = K \text{ x } A$

As the dimensions of area, A, are L^2, the above expression becomes

$F_e = K \text{ x } L^2$

The ratio between the inertia force, F_i, and the force due to elasticity, F_e, is given by

$F_i / F_e = (\rho L^2 V^2) / (KL^2) = \rho V^2 / K = V^2 / (K / \rho) = V^2 / C^2$

where, $C = (K / \rho)^{1/2}$ = velocity of sound in that fluid medium whose bulk modulus of elasticity, K, and mass density, □, are being considered.
The ratio (V^2 / C^2) is known as the ***'Cauchy number'***. The square root of this ratio, i.e., (V / C) or $\{V / (K / \rho)^{1/2}\}$ is known as the ***'Mach number'***, *Ma* or N_M. When $Ma > 1$, i.e., $V > C$, or in other words, the characteristic velocity of flowof the fluid is more than velocity of sound in that flow medium, the flow is said to be supersonic. When $Ma < 1$, i.e., $V < C$, or in other words, the characteristic velocity of flow of the fluid is less than velocity of sound in that flow medium, the flow is said to be subsonic. When $Ma = 1$, or $V = C$, the flow is considered to be sonic. When $Ma >> 1$, i.e., $V >> C$, then the flow is sometimes termed as hypersonic. A higher

Mach number indicates the predominance of the effect of compressibility of the fluid. However, when the Mach number is relatively small, say, less than 0.4, the effect of compressibility of the fluid can be neglected.

(e) Inertia force – Surface tension force ratio (Weber number)

Force due to surface tension, $F_s = \sigma L$

where □ = surface tension of fluid in contact with, say, air (in N/m)
L = length of the fluid film over which the force due to surface tensionacts
Hence, the ratio of the inertia force, F_i, and the surface tension force, F_s, isgiven by

$$F_i / F_s = (\rho L^2 V^2) / (\sigma L) = (\rho L V^2) / \sigma = [V^2 / \{\sigma / (\rho L)\}]$$

The square root of this ratio, i.e., $[V / \{\sigma / (\rho L)\}^{1/2}]$ is called the ***Weber number***.

SIMILARITY LAWS OR MODEL LAWS

The results obtained from the model tests can be transferred to the prototype by the use of model laws. The model laws can be developed from the principles of dynamic similarity. The conditions for the existence of dynamic similarity between the model and the prototype are depicted by equations (1) to (6). In almost all hydraulic problems encountered in practice, for which model studies are required to be carried out, it is quite rare that all the forces, namely, F_i, F_g, F_v, F_p, F_e and F_s are simultaneously predominant in the flow phenomenon. Moreover, in most of the fluid flow problems, only one force in addition to the inertia force, F_i, is relatively more significant than the rest of the forces. The rest of the forces may either do not exist or may be of negligible magnitude. Under these circumstances, the various model laws have been developed depending upon the significant influence of each of the forces on the different fluid flow phenomena. In the derivation of these model laws, it has been assumed that for equal values of the dimensionless parameters the corresponding flow pattern in model and its prototype are similar.

(a) Reynolds Model Law

In case of flows where, in addition to the inertia force, the only other force of significance is the viscous force, the similarity in flow in the model and the prototype can be obtained if the Reynolds number of flow is the same in both the model and the prototype. This is known as ***Reynolds Model Law***.

According to the law, we have, $(N_R)_{model} = (N_R)_{prototype}$

$(\rho_m V_m L_m) / \mu_m = (\rho_p V_p L_p) / \mu_p$

where $(N_R)_{model}$ = Reynolds number of flow in model $(N_R)_{prototype}$ = Reynolds number of flow in prototype

ρ_m = mass density of fluid in model

V_m = characteristic velocity of flow in model

L_m = characteristic length in model

μ_m = dynamic viscosity of fluid in model

ρ_p = mass density of fluid in prototype

V_p = characteristic velocity of flow in prototype

L_p = characteristic length in prototype

μ_p = dynamic viscosity of fluid in prototype

Dividing LHS by RHS of above equation,

$[(\rho_m V_m L_m) / \mu_m] / [(\rho_p V_p L_p) / \mu_p] = (\rho_m / \rho_p)(V_m / V_p)(L_m / L_p)(\mu_p / \mu_m)$

$= \{(\rho_m / \rho_p)(V_m / V_p)(L_m / L_p)\} / \{(\mu_m / \mu_p)\}$

$= \rho_r V_r L_r / \mu_r = 1$ …… (7)

where ρ_r = Mass density scale ratio

V_r = characteristic velocity scale ratioL_r = Length scale ratio

μ_r = dynamic viscosity scale ratio

Equation (7) may be used to obtain the scale ratios for various other physicalquantities on the basis of Reynolds model law.

Let us derive the scale ratios for models of certain quantities governed byReynolds model law.

Scale ratio for Velocity (V_r):

From equation (7), $V_r = \mu_r / (\rho_r L_r)$ …… (8)

Scale ratio for time (T_r):

The scale ratio for velocity can be written as $V_r = V_m / V_p$

$= (L_m / T_m) / (L_p / T_p)$

$= (L_m / L_p) (T_p / T_m)$

$= (L_m / L_p) \{1 / (T_m / T_p)\}$

$= L_r / T_r$

Substituting $V_r = L_r / T_r$ in equation (7), we have,

$\{_r (L_r / T_r) L_r\} / \mu_r = 1$

$\rho_r L_r{}^2 / \mu T = 1$

$T_r = \rho_r \ L^2 / \mu_r$ …… (9)

Scale ratio for acceleration (a_r):

Acceleration scale ratio, $a_r = a_m / a_p = \{L_m / (T_m)^2\} / \{L_p / (T_p)^2\}$

$= (L_m / L_p) \{(T_p)^2 / (T_m)^2\}$

$= (L_m / L_p) [1 / \{(T_m)^2 / (T_p)^2\}]$

$= L_r / T_r{}^2$

From equation (7), we have,

$V_r = \mu_r / (\rho_r L_r)$

$a_r = V_r / T_r = \{ \mu_r / (\rho_r L_r)\} / \{ \rho L_r{}^2 / \mu_r\}$

Putting the expression for T_r from equation (9) in the above expression, wehave,

$a_r = \{\mu_r / (\rho_r L_r)\} / T_r = = \mu_r{}^2 / \rho^2 L_r{}^3$ …… (10)

Scale ratio for discharge (Q_r):

We know that, $Q_r = A_r V_r = L_r{}^2 V_r$

From equation (7), we have,$V_r = \mu_r / (\rho_r L_r)$

Hence, $Q = L_r^2 \{ \mu_r / (\rho_r L_r)\} = L_r \mu_r / \rho_r$ …… (11)

Scale ratio for force (F_r):

Shear force due to viscosity of fluid = shear stress x area

$= \tau A = (V/y) A\backslash$

$F_r = \mu_r (V_r/y_r)A_r = \mu_r (V_r/L_r) (L^2) = \mu_r V_r L_r$

From equation (7), we have, $V_r = \mu_r / (\rho_r L_r)$

Putting $V_r = \mu_r / \rho_r L_r)$ in the above expression for F_r

$F_r = \mu_r \{ \mu_r / \rho_r L_r)\} L_r = \mu^2 /\rho_r$ …… (12)

Some of the phenomena for which Reynolds model law can be a sufficient criterion for similarity of flow in the model and the prototype are:

(i) flow of incompressible fluid in closed pipes

(ii) motion of submarines completely under water

(iii) motion of air planes

(iv) flow around structures and other bodies immersed completely under moving fluids

(b) Froude Model Law

In case of flows where, in addition to the inertia force, the only other force of significance is the force of gravity, the similarity in flow in the model and the prototype can be obtained if the Froude number of flow is the same in both the model and the prototype. This is known as ***Froude Model Law***.

According to the law, we have, $(Fr)_{model} = (Fr)_{prototype}$

$V_m / (L_m g_m)^{1/2} = V_p / (L_p g_p)^{1/2}$ ...... (13)

where $(Fr)_{model}$ = Froude number of flow in model $(Fr)_{prototype}$ = Froude number of flow in prototypeV_m = velocity of flow in model

L_m = characteristic dimension (length) in model

g_m = acceleration due gravity at the site of model testingV_p = velocity of flow in prototype

L_p = characteristic dimension (length) in prototype g_p = acceleration due gravity at the site of prototype

Dividing LHS by RHS of above equation,

$V_m / (L_m g_m)^{1/2} / V_p / (L_p g_p)^{1/2} = 1$

$V_r / (g_r L_r)^{1/2} = 1$

$V_r = (g_r L_r)^{1/2}$ ...... (14)

Since in most cases, as the value of g at the site of model testing will practically be the same as the value of g at the site of the proposed prototype, we have the scale ratio of g, i.e., $g_r = g_m / g_p = 1$

Hence, equation (14) becomes

$V_r = L_r^{1/2}$

$V_r / L_r^{1/2} = 1$ ...... (15)

Equation (14) or (15) may be used to obtain the scale ratios for various other physical quantities.

Let us derive the scale ratios for models of certain quantities governed by Froude model law.

Scale ratio for Time (T_r):

As discussed previously, the scale ratio for velocity can be written as$V_r = L_r / T_r$

Substituting $V_r = L_r / T_r$ in equation (14), we have,

$L_r / T_r = (g_r L_r)^{1/2}$

$Tr = Lr / (grLr)^{1/2} = Lr^{1/2} / gr^{1/2}$ ...... (16)

Scale ratio for acceleration (a_r):

As discussed previously, acceleration scale ratio,

$a_r = L_r / T_r^2$

We have just expressed the scale ratio for time, T_r, as

$Tr = Lr^{1/2} / gr^{1/2}$

Putting the above expression for T_r in $a_r = L_r / T_r^2$, we have,

$a_r = L_r / (L_r^{1/2} / g_r^{1/2})^2 = L_r / (L_r / g_r) = g_r$ ...... (17)

Scale ratio for discharge (Q_r):

We know that, $Q_r = A_r V_r = L^2 V_r$

From equation (14), we have, $V_r = (g_r L_r)^{1/2}$

Substituting the above expression for V_r in the expression for Q_r stated above,we have,

$Q_r = L^2 (g_r L_r)^{1/2} = L^{5/2} g^{1/2}$ …… (18)

Scale ratio for force (F_r):

Force due to gravity (weight of fluid) = mass x acceleration due to gravity

= Mg

where M = mass of fluid

g = acceleration due to gravity

M = mass density of fluid x volume of fluid =ρ x volume of fluid = ρL^3

Hence, $F = \rho L^3 g$

So, Scale ratio for force, $F_r = \rho_r L^3 g_r$ ...... (19)

Some of the phenomena for which Reynolds model law can be a sufficient criterion for dynamic similarity of flow in the model and the prototype are:

(i) Free-surface flows such as flow over spillways, sluices, etc.,

(ii) Flow of jet from an orifice or nozzle

(iii) Problems in which waves are likely to be formed on the surface

(iv) Problems in which fluids of different densities flow over one another

(c) Euler Model Law

In case of fluid systems where, in addition to the inertia force, the only other force of significance is the force due to supplied pressures, the dynamic similarity in flow in the model and the prototype can be obtained if the Euler number of flow is the same in both the model and the prototype. This is knownas ***Euler Model Law***.

$(Eu)_{model} = (Eu)_{prototype}$

$[V_m / (p_m/\rho_m)^{1/2}] = [V_p / (p_p/\rho_p)^{1/2}]$ ...... (20)

where $(Eu)_{model}$ = Euler number of flow in model

$(Eu)_{prototype}$ = Euler number of flow in prototype

V_m = velocity of flow in the model

p_m = intensity of fluid pressure in the model

ρ_m = mass density of fluid in the model

V_p = velocity of flow in the prototype

p_p = intensity of fluid pressure in the prototype

ρ_p = mass density of fluid in the prototype

Dividing LHS by RHS of above equation,

$[V_m / (p_m/\rho_m)^{1/2}] / [V_p / (p_p/\rho_p)^{1/2}] = 1$

$[V_r / (p_r/\rho_r)^{1/2}] = 1$ ...... (21)

Equation (21) represents the Euler Model Law which may be used to evaluatescale ratios for various other physical quantities.

Euler model law may be considered as an essential requirement for establishing dynamic similarity in an enclosed fluid system where the turbulence is fully developed and the viscous forces are insignificant, and also the forces of gravityand surface tension are completely absent.

(d) Mach Model Law

In case of fluid flow phenomena where, in addition to the inertia force, the only other force of significance is the force resulting from elastic compression, the dynamic similarity in flow in the model and the prototype can be obtained if the Mach number of flow is the same in both the model and the prototype. This is known as ***Mach Model Law***.

$(Ma)_{model} = (Ma)_{prototype}$

$[V_m / (K_m / \rho_m)^{1/2}] = [V_p / (K_p / \rho_p)^{1/2}]$ ...... (22)

where $(Ma)_{model}$ = Mach number of flow in model

$(Ma)_{prototype}$ = Mach number of flow in prototype

V_m = velocity of flow in the model

K_m = bulk modulus of elasticity of fluid in the model

ρ_m = mass density of fluid in the model

V_p = velocity of flow in the prototype

K_p = bulk modulus of elasticity of fluid in the prototype

ρ_p = mass density of fluid in the prototypeDividing LHS by RHS of above equation,

$[V_m / (K_m / \rho_m)^{1/2}] / [V_p / (K_p / \rho_p)^{1/2}] = 1$

$[V_r / (K_r / \rho_r)^{1/2}] = 1$ ...... (23)

Equation (23) represents the Mach Model Law which may be used to evaluate scale ratios for various other physical quantities.

The Mach model law finds extensive application in aerodynamic testing and in phenomena involving velocities exceeding the speed of sound. It is also applicable in hydraulic model testing for cases of unsteady flow, especially water hammer problems.

(e) Weber Model Law

In case of fluid flow phenomena where, in addition to the inertia force, the only other force of significance is the force resulting from surface tension, the dynamic similarity in flow in the model and the prototype can be obtained if theWeber number of flow is the same in both the model and the prototype. This is known as ***Weber Model Law***.

$(We)_{model} = (We)_{prototype}$

$[V_m / \{\sigma_m / (\rho_m L_m)\}^{1/2}] = [V_p / \{\sigma_p / (\rho_p L_p)\}^{1/2}]$ …… (24)

where $(We)_{model}$ = Weber number of flow in model

$(We)_{prototype}$ = Weber number of flow in prototype

V_m = velocity of flow in the model

σ_m = surface tension of fluid in the model

ρ_m = mass density of fluid in the modelL_m = characteristic length in the modelV_p = velocity of flow in the prototype

σ_p = surface tension of fluid in the prototype

ρ_p = mass density of fluid in the prototype

L_p = characteristic length in the prototypeDividing LHS by RHS of above equation,

$[V_m / \{\sigma_m / (\rho_m L_m)\}^{1/2}] / [V_p / \{\sigma_p / (\rho_p L_p)\}^{1/2}] = 1$

$[V_r / \{\sigma_r / (\rho_r L_r)\}^{1/2}] = 1$ …… (25)

Equation (25) represents the Weber Model Law which may be used to evaluatescale ratios for various other physical quantities.

Weber model law can be applied in the following cases:

(i) flow over weirs involving very low heads

(ii) very thin sheet of liquid flowing over a surface

(iii) capillary waves in channels