Properties of Group:

1). The identity element in a group is unique.

2). The inverse element in a group is unique

3). Cancellation law:

$\forall a, b \in G$ i). $a * b = a * c \Rightarrow b = c$ [Left cancellation]

ii). $b * a = c * a$ [Right cancellation] $\Rightarrow b = c$

4). Let G be a group.

If $a, b \in G$ then $(a * b)^{-1} = b^{-1} * a^{-1}$

Proof:

Let $a, b \in G$

and a^{-1}, b^{-1} be their inverses resly.

Let $(G, *)$ be a group.

Let $a, b \in G$. $a * a^{-1} = a^{-1} * a = e$

$b * b^{-1} = b^{-1} * b = e$

Now $(a * b) * (b^{-1} * a^{-1}) = a * (b * b^{-1}) * a^{-1}$ (Associative)

$= (a * e) * a^{-1}$

$= a * a^{-1}$ Identity

$(a * b) * (b^{-1} * a^{-1}) = e \rightarrow (1)$

IIIy $(b^{-1} * a^{-1}) * (a * b) = e \rightarrow (2)$

From (1) and (2), $(a * b)^{-1} = b^{-1} * a^{-1}$

The inverse of $(a * b)$ is $b^{-1} * a^{-1}$

Hence proved.

5). Prove that a group $(G, *)$ is abelian iff $(a * b)^2 = a^2 * b^2, \forall a, b \in G$

Proof:

Assume that G is abelian

$\therefore a * b = b * a, \quad a, b \in G \rightarrow (1)$

Now $a^2 * b^2 = (a*a) * (b*b)$
$= a * [a * (b*b)]$ Associative
$= a * [(a*b) * b]$ Associative
$= a * [(b*a) * b]$ By (1)
$= (a*b) * (a*b)$ Associative
$= (a*b)^2$

conversely,
Assume that $(a*b)^2 = a^2 * b^2$
$(a*b) * (a*b) = (a*a) * (b*b)$
$a * [b * (a*b)] = a * [a * (b*b)]$
$b * (a*b) = a * (b*b)$ left cancellation law
$(b*a) * b = (a*b) * b$
$b*a = a*b$ Right cancellation law
$\Rightarrow$ G is abelian.

6). If $(G, *)$ is an abelian group, ST
$(a*b)^n = a^n * b^n$, $\forall a, b \in G$, where n is a +ve integer
proof:
Since $(G, *)$ is abelian, we've
$a*b = b*a, \quad \forall a, b \in G \longrightarrow (1)$
For $a, b \in G$, we've $(a*b)^1 = (b*a)^1$ by (1)
and $(a*b)^2 = (a*b) * (a*b)$
$= a * (b*a) * b$ Associative
$= a * (a*b) * b$ by (1)
$= (a*a) * (b*b)$ Associative
$= a^2 * b^2$
Thus, the result is true for $n = 1, 2, \ldots$
Let us assume that the result is valid for $n = m$.

$$(a*b)^m = a^m * b^m \quad \rightarrow (2)$$

Now $(a*b)^{m+1} = (a*b)^m * (a*b)$

$= (a^m * b^m) * (a*b)$ By (2)

$= a^m * (b^m * a) * b$ Associative

$= a^m * (a * b^m) * b$ (G is abelian)

$= (a^m * a) * (b^m * b)$

$(a*b)^{m+1} = a^{m+1} * b^{m+1}$

Hence by induction, the result is true for +ve integral values of n.

7]. In a group G, Prove that an elt. $a \in G$ such that $a^2 = e$, $a \neq e$ iff $a = a^{-1}$

Proof:

Assume that $a = a^{-1}$

To prove: $a^2 = e$

Now consider $a^2 = a * a$

$= a * a^{-1}$

$= e$

Conversely, Assume $a^2 = e$

To prove: $a = a^{-1}$

Now $a^2 = e$

$a * a = e$

$a^{-1} * (a*a) = a^{-1} * e$ [Premultiply by a^{-1}]

$(a^{-1} * a) * a = a^{-1}$

$e * a = a^{-1}$

$a = a^{-1}$.

8]. If ery. elt. of a group G has its own inverse, then G is abelian.

Proof:

Let $(G, *)$ be a group.

for $a, b \in G$, we've $a * b \in G$

Given $a = a^{-1}$ and $b = b^{-1}$

Now $a * b = (a * b)^{-1}$ [It has its own inverse]

$= b^{-1} * a^{-1}$ (By property)

$= b * a$

$\therefore G$ is abelian

Converse need not be true.