



DEPARTMENT OF MATHEMATICS

Properties of lattices:

Let (L, \wedge, \vee) be a given lattice. Then for any $a, b, c \in L$.

1). Idempotent law

$$a \wedge a = a \text{ and } a \vee a = a$$

2). Commutative law

$$a \wedge b = b \wedge a \text{ and } a \vee b = b \vee a$$

3). Associative law

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \text{ and } (a \vee b) \vee c = a \vee (b \vee c)$$

4). Absorption law

$$a \wedge (a \vee b) = a \text{ and } a \vee (a \wedge b) = a$$

Proof:

1). Idempotent law:

$$\text{Now } a \vee a = \text{LUB}(a, a) = \text{LUB}(a) = a$$

$$a \vee a = a$$

$$\text{and } a \wedge a = \text{GLB}(a, a) = \text{GLB}(a) = a$$

$$a \wedge a = a$$

2). Commutative law:

$$\text{Now } a \vee b = \text{LUB}(a, b) = \text{LUB}(b, a) = b \vee a$$

$$\text{and } a \wedge b = \text{GLB}(a, b) = \text{GLB}(b, a) = b \wedge a$$

3). Associative:

$$\text{Let } a \vee (b \vee c) = d \rightarrow (1)$$

$$(a \vee b) \vee c = e \rightarrow (2)$$

$$(1) \Rightarrow d \text{ is LUB of } (a, b \vee c)$$

$$\Rightarrow d \geq a \text{ and } d \geq b \vee c \rightarrow (3)$$

$$\text{WKT } b \vee c \text{ is LUB of } (b, c)$$

$$b \vee c \geq b \text{ and } b \vee c \geq c \rightarrow (4)$$

$$\left. \begin{array}{l} d \geq a \\ d \geq b \\ d \geq c \end{array} \right\}$$

$$\rightarrow (5)$$



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(5) $\Rightarrow d$ is an UB of (a, b) and $d \geq c$
 $d \geq \text{LUB}(a, b)$
 $d \geq a \vee b$ and $d \geq c$

$\therefore d$ is an UB of $(a \vee b, c)$

Since e is LUB of $(a \vee b, c)$

$$d \geq e$$

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$$e \geq d$$

$$\therefore e = d \Rightarrow a \vee (b \vee c) = (a \vee b) \vee c$$

4). Absorption law:

$$a \vee (a \wedge b) = a$$

Since $a \wedge b$ is GILB of $\{a, b\}$

$$a \wedge b \leq a \rightarrow (1)$$

obviously $a \leq a \rightarrow (2)$

(1) and (2), $a \vee (a \wedge b) \leq a \rightarrow (3)$

By defn of LUB, we've

$$a \leq a \vee (a \wedge b) \rightarrow (4)$$

$$a \vee (a \wedge b) = a.$$

Theorem: Isotonicity of law or property

Let (L, \wedge, \vee) be a gm lattice for any $a, b, c \in L$. then prove that

$$b \leq c \Rightarrow \begin{cases} 1). a \wedge b \leq a \wedge c \\ 2). a \vee b \leq a \vee c \end{cases}$$

Proof:

Given $b \leq c$

GILB $\{b, c\} = b \wedge c = b$ (since $b \leq c \Leftrightarrow b \vee c = c$)
 LUB $\{b, c\} = b \vee c = c$ $\Leftrightarrow b \wedge c = b$

To prove:

1). $a \wedge b \leq a \wedge c$.

It is enough to prove that $\text{GILB}(a \wedge b, a \wedge c) = a \wedge b$

i.e., $(a \wedge b) \wedge (a \wedge c) = a \wedge b$



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Now

$$\begin{aligned}
 (a \wedge b) \wedge (a \wedge c) &= a \wedge (b \wedge a) \wedge c \\
 &= a \wedge (a \wedge b) \wedge c \\
 &= (a \wedge a) \wedge (b \wedge c) \\
 &= a \wedge (b \wedge c) \\
 &= a \wedge b
 \end{aligned}$$

$$\therefore a \wedge b \leq a \wedge c$$

$$2). \quad a \vee b \leq a \vee c$$

It is enough to prove $\text{LUB}(a \vee b, a \vee c) = a \vee c$

$$(a \vee b) \vee (a \vee c) = a \vee c$$

Now

$$\begin{aligned}
 (a \vee b) \vee (a \vee c) &= a \vee (b \vee a) \vee c \\
 &= a \vee (a \vee b) \vee c \\
 &= (a \vee a) \vee (b \vee c) \\
 &= a \vee (b \vee c) \\
 &= a \vee c
 \end{aligned}$$

$$\therefore a \vee b \leq a \vee c$$

Theorem: Distributive Inequality

Let (L, \wedge, \vee) be a given lattice for any $a, b, c \in L$, the following inequality holds.

$$i). \quad a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$

$$ii). \quad a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$$

Proof:

$$i). \quad a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$

From the defn. of LUB, It is obvious that

$$a \leq a \vee b \quad \rightarrow (1)$$

$$\text{and } b \wedge c \leq b \leq a \vee b$$

$$\Rightarrow b \wedge c \leq a \vee b \quad \rightarrow (2)$$



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From (1) and (2),

$a \vee b$ is an upper bound of $\{a, b \wedge c\}$

Hence $a \vee b \geq a \vee (b \wedge c) \rightarrow (A)$

From the defn. of LUB, it is obvious that

$a \leq a \vee c \rightarrow (B)$

and $b \wedge c \leq c \leq a \vee c$

$\Rightarrow b \wedge c \leq a \vee c \rightarrow (4)$

From (3) and (4), $a \vee c$ is an UB of $\{a, b \wedge c\}$

Hence $a \vee c \geq a \vee (b \wedge c) \rightarrow (B)$

From (A) and (B), we've $a \vee (b \wedge c)$ is a LB of $\{a \vee b, a \vee c\}$.

$\therefore a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Hence proved (i).

ii). $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

wkt $a \geq a \wedge b \rightarrow (1)$

and $b \vee c \geq b \geq a \wedge b$

$b \vee c \geq a \wedge b \rightarrow (2)$

From (1) and (2), $a \wedge b$ is an LB of $\{a, b \vee c\}$

Hence $a \wedge b \leq a \wedge (b \vee c) \rightarrow (C)$

wkt

$a \geq a \wedge c \rightarrow (3)$

and $b \vee c \geq c \geq a \wedge c$

$b \vee c \geq a \wedge c \rightarrow (4)$

From (3) & (4), $a \wedge c$ is an LB of $\{a, b \vee c\}$

Hence $a \wedge c \leq a \wedge (b \vee c) \rightarrow (D)$

From (C) and (D), we've $a \wedge (b \vee c)$ is an UB of $\{a \wedge b, a \wedge c\}$.

$\therefore a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Hence proved (ii).

