Properties of lattices:

Let $(L, \wedge, \vee)$ be a given lattice. Then for any $a, b, c \in L$.

1). Idempotent law

$a \wedge a = a$ and $a \vee a = a$

2). Commutative law

$a \wedge b = b \wedge a$ and $a \vee b = b \vee a$

3). Associative law

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$

4). Absorption law

$a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

Proof:

1). Idempotent law:

Now $a \vee a = LUB(a, a) = LUB(a) = a$

$a \vee a = a$

and $a \wedge a = GLB(a, a) = GLB(a) = a$

$a \wedge a = a$

2). Commutative law:

Now $a \vee b = LUB(a, b) = LUB(b, a) = b \vee a$

and $a \wedge b = GLB(a, b) = GLB(b, a) = b \wedge a$

3). Associative:

Let $a \vee (b \vee c) = d \rightarrow (1)$

$(a \vee b) \vee c = e \rightarrow (2)$

$(1) \Rightarrow d$ is LUB of $(a, b \vee c)$

$\Rightarrow d \geq a$ and $d \geq b \vee c \rightarrow (3)$

WKT $b \vee c$ is LUB of (b, c)

$b \vee c \geq b$ and $b \vee c \geq c \rightarrow (4)$

$$\left.\begin{array}{l} d \geq a \\ d \geq b \\ d \geq c \end{array}\right\} \rightarrow (5)$$

(5) $\Rightarrow$ d is an UB of (a, b) and $d \succeq c$

$d \succeq LUB(a, b)$

$d \succeq a \vee b$ and $d \succeq c$

$\therefore$ d is an UB of $(a \vee b, c)$

Since e is LUB of $(a \vee b, c)$

$d \succeq e$

Illy $e \succeq d$

$\therefore$ $e = d \Rightarrow a \vee (b \vee c) = (a \vee b) \vee c$

4). Absorption law:

$a \vee (a \wedge b) = a$

Since $a \wedge b$ is GLB of $\{a, b\}$

$a \wedge b \leq a \rightarrow (1)$

obviously $a \leq a \rightarrow (2)$

(1) and (2), $a \vee (a \wedge b) \leq a \rightarrow (3)$

By defn. of LUB, we've

$a \leq a \vee (a \wedge b) \rightarrow (4)$

$a \vee (a \wedge b) = a$.

Theorem: Isotonicity of law or property

Let $(L, \wedge, \vee)$ be a gvn. lattice for any $a, b, c \in L$. then prove that

$b \leq c \Rightarrow \begin{cases} 1). & a \wedge b \leq a \wedge c \\ 2). & a \vee b \leq a \vee c \end{cases}$

Proof:

Given $b \leq c$

$GLB\{b, c) = b \wedge c = b$ (Since $b \leq c \Leftrightarrow b \vee c = c$

$LUB(b, c) = b \vee c = c$ $\Leftrightarrow b \wedge c = b$)

To prove:

1). $a \wedge b \leq a \wedge c$.

It is enough to prove that $GLB(a \wedge b, a \wedge c) = a \wedge b$

i.e., $(a \wedge b) \wedge (a \wedge c) = a \wedge b$

Now

$$(a\wedge b)\wedge(a\wedge c) = a\wedge(b\wedge a)\wedge c$$
$$= a\wedge(a\wedge b)\wedge c$$
$$= (a\wedge a)\wedge(b\wedge c)$$
$$= a\wedge(b\wedge c)$$
$$= a\wedge b$$

$\therefore a\wedge b \le a\wedge c$

2). $a\vee b \le a\vee c$

It is enough to prove $LUB(a\vee b, a\vee c) = a\vee c$

$$(a\vee b)\vee(a\vee c) = a\vee c$$

Now

$$(a\vee b)\vee(a\vee c) = a\vee(b\vee a)\vee c$$
$$= a\vee(a\vee b)\vee c$$
$$= (a\vee a)\vee(b\vee c)$$
$$= a\vee(b\vee c)$$
$$= a\vee c$$

$\therefore a\vee b \le a\vee c$

Theorem: Distributive inequality

Let $(L, \wedge, \vee)$ be a given lattice for any $a, b, c \in L$, the following inequality holds.

i). $a\vee(b\wedge c) \le (a\vee b)\wedge(a\vee c)$

ii). $a\wedge(b\vee c) \ge (a\wedge b)\vee(a\wedge c)$

Proof:

i). $a\vee(b\wedge c) \le (a\vee b)\wedge(a\vee c)$

From the defn. of LUB, it is obvious that

$$a \le a\vee b \longrightarrow (1)$$

and $b\wedge c \le b \le a\vee b$

$$\Rightarrow b\wedge c \le a\vee b \longrightarrow (2)$$

From (1) and (2),
$a \vee b$ is an upper bound of $\{a, b \wedge c\}$
Hence $a \vee b \geq a \vee (b \wedge c) \rightarrow (A)$
From the defn. of LUB, it is obvious that
$a \leq a \vee c \rightarrow (3)$
and $b \wedge c \leq c \leq a \vee c$
$\Rightarrow b \wedge c \leq a \vee c \rightarrow (4)$
From (3) and (4), $a \vee c$ is an UB of $\{a, b \wedge c\}$
Hence $a \vee c \geq a \vee (b \wedge c) \rightarrow (B)$
From (A) and (B), we've $a \vee (b \wedge c)$ is a LB of $\{a \vee b, a \vee c\}$.
$\therefore a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
Hence proved (i).

ii). $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$
WKT $a \geq a \wedge b \rightarrow (1)$
and $b \vee c \geq b \geq a \wedge b$
$b \vee c \geq a \wedge b \rightarrow (2)$
From (1) and (2), $a \wedge b$ is an LB of $\{a, b \vee c\}$
Hence $a \wedge b \leq a \wedge (b \vee c) \rightarrow (C)$
WKT
$a \geq a \wedge c \rightarrow (3)$
and $b \vee c \geq c \geq a \wedge c$
$b \vee c \geq a \wedge c \rightarrow (4)$
From (3) & (4), $a \wedge c$ is an LB of $\{a, b \vee c\}$ $\rightarrow (D)$
Hence $a \wedge c \leq a \wedge (b \vee c)$
From (c) and (d), we've $a \wedge (b \vee c)$ is an UB of $\{a \wedge b, a \wedge c\}$.
$\therefore a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$
Hence proved (ii).