PROPERTIES OF GROUPS:

property 1: The identity element of a group is unique.

proof: Let $(G, *)$ be a group.

Let e_1 and e_2 be two identity elements in G.

Suppose e_1 is the identity, then

$$e_1 * e_2 = e_2 * e_1 = e_2 \quad \text{——} ①$$

Suppose e_2 is the identity, then

$$e_2 * e_1 = e_1 * e_2 = e_1 \quad \text{——} ②$$

From ① & ② we get

$$e_1 = e_2,$$

$\therefore$ The identity element is unique.

property 2: In a group $(G, *)$, the left and right cancellation laws are true. i.e. $a * b = a * c \Rightarrow b = c$ [left cancellation]

$b * a = c * a \Rightarrow b = c$ [Right cancellation]

proof:

Let $(G, *)$ be a group.

Let $a \in G$ and hence $a^{-1} \in G$

Then $a * a^{-1} = a^{-1} * a = e$

(i) Left cancellation law:

Let $a * b = a * c$

pre multiply by a^{-1} on both sides, we get,

$$a^{-1} * (a * b) = a^{-1} * (a * c)$$
$$(a^{-1} * a) * b = (a^{-1} * a) * c$$
$$e * b = e * c$$
$$b = c$$

(ii) Right cancellation law:

Let $b * a = c * a$

post multiply by a^{-1} on both sides, we get,

$$(b * a) * a^{-1} = (c * a) * a^{-1}$$
$$b * (a * a^{-1}) = c * (a * a^{-1})$$
$$b * e = c * e$$
$$b = c$$

Property 3: The inverse element of a group is unique

Proof: Let $(G, *)$ be a group.

Let $a \in G$ and e be the identity of G.

Let a_1^{-1} and a_2^{-1} be the two different inverses of the same element.

$$a_1^{-1} * a = a * a_1^{-1} = e$$
$$a_2^{-1} * a = a * a_2^{-1} = e$$

$$(a_1^{-1} * a) * a_2^{-1} = e * a_2^{-1} = a_2^{-1} \quad \text{—— (1)}$$

$$a_1^{-1} * (a * a_2^{-1}) = a_1^{-1} * e = a_1^{-1} \quad \text{—— (2)}$$

From (1) & (2) we get, $a_1^{-1} = a_2^{-1}$

$\therefore$ There will be no two different inverses for the same element.

Property 4: In a group $(a^{-1})^{-1} = a$, $a \in G$ (ie) the inverse of a^{-1} is a.

Proof:

Let $(G, *)$ be a group.

Let 'e' be the identity element.

WKT, $a^{-1} * a = e = a * a^{-1}$, $a \in G$

$$(a^{-1})^{-1} * (a^{-1} * a) = (a^{-1})^{-1} * e$$
$$= (a^{-1})^{-1} \quad \text{—— (1)}$$

But $((a^{-1})^{-1} * a^{-1}) * a = e * a$
$$= a \quad \text{—— (2)}$$

From (1) & (2) we get, $(a^{-1})^{-1} = a$

Note: The above property is involution law.

Property 5: Let G be a group. If $a, b \in G$ then $(a * b)^{-1} = b^{-1} * a^{-1}$ (or) The inverse of the product of two element is equal to the product of their inverses in reverse order.

proof: Let $a, b \in G$ and a^{-1}, b^{-1} be their inverses respectively.

$\therefore a * a^{-1} = a^{-1} * a = e$

and $b * b^{-1} = b^{-1} * b = e$

$$(a*b)*(b^{-1}*a^{-1}) = a*[b*(b^{-1}*a^{-1})] \quad [* \text{ is associative}]$$
$$= a*[(b*b^{-1})*a^{-1}]$$
$$= a*[e*a^{-1}]$$
$$= a*a^{-1}$$
$$= e \quad \text{——} (1)$$

Similarly we can prove that

$$(b^{-1}*a^{-1})*(a*b) = e \quad \text{——} (2)$$

From (1) & (2) we get

$$(a*b)*(b^{-1}*a^{-1}) = e \Rightarrow (a*b)*(a*b)^{-1} = e$$
$$\Rightarrow (a*b)^{-1} = b^{-1}*a^{-1}$$

$\therefore$ The inverse of $a*b$ is $b^{-1}*a^{-1}$.

<u>property 6</u>: For any group G, if $a^2 = e$ with $a \neq e$ then G is abelian (or) if every element of a group G has its own inverse, then G is abelian. Is the converse true.

<u>proof</u>: Let $(G, *)$ be a group.

For $a, b \in G$, we have $a * b \in G$

Given $a = a^{-1}$ and $b = b^{-1}$

$$a * b = (a*b)^{-1} \quad \text{[By property 5]}$$
$$= b^{-1} * a^{-1}$$
$$= b * a \quad [\text{Given } a = a^{-1};\ b = b^{-1}]$$

$\therefore$ G is abelian.

The converse need not be true.

Property 7: Prove that a group $(G, *)$ is abelian iff $(a*b)^2 = a^2 * b^2 \ \forall a, b \in G$

Proof: Assume that G is abelian.

$$a * b = b * a, \ a, b \in G$$

$$a^2 * b^2 = (a*a) * (b*b)$$
$$= a * [a * (b*b)]$$
$$= a * [(a*b) * b]$$
$$= a * [(b*a) * b]$$
$$= (a*b) * (a*b)$$
$$= (a*b)^2$$

$\therefore (a*b)^2 = a^2 * b^2$.

Conversely, assume that,

$$(a*b)^2 = a^2 * b^2$$
$$(a*b)*(a*b) = (a*a)*(b*b)$$
$$a*[b*(a*b)] = a*[a*(b*b)]$$
$$b*(a*b) = a*(b*b)$$
$$(b*a)*b = (a*b)*b$$
$$b*a = a*b$$

$\therefore$ G is abelian.

Example:

① prove that in an abelian group $(ab)^2 = a^2b^2$.

Soln:
$$(ab)^2 = (ab)(ab)$$
$$= a(ba)b$$
$$= a(ab)b$$
$$= (aa)(bb)$$
$$= a^2b^2$$

② In a group G prove that an element $a \in G \ni a^2 = e$, $a \neq e$ iff $a = a^{-1}$.

Soln: Assume that $a = a^{-1}$

To prove: $a^2 = e$

Now consider $a^2 = a * a$

$= a * a^{-1}$

$= e$

$\therefore \; a^2 = e.$

Conversely assume $a^2 = e$

<u>To prove</u>: $a = a^{-1}$.

Now consider $a^2 = e$

$a * a = e$

$a^{-1} * (a * a) = a^{-1} * e$ [premultiply by a^{-1}]

$(a^{-1} * a) * a = a^{-1}$

$e * a = a^{-1}$

$a = a^{-1}$