DEFINITION:

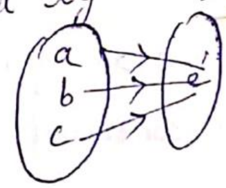
MORPHISM OF GROUPS:

Let $(G,*)$ and (H,Δ) be any two groups. A mapping $f: G \to H$ is said to be a homomorphism, if $f(a*b) = f(a)\,\Delta\, f(b)$ for any $a,b \in G$.

KERNEL OF A HOMOMORPHISM:

Let $f: G \to G'$ be a group homomorphism. The set of elements of G which are mapped into e' (identity in G') is called the kernel of f and it is denoted by $\ker(f)$.

$$\ker(f) = \{x \in G \mid f(x) = e'\}$$



ISOMORPHISM:

A mapping f from a group $(G,*)$ to a group (G',Δ) is said to be an isomorphism if

(i) f is a homomorphism, $f(a*b) = f(a)\,\Delta\, f(b)$, for all $a,b \in G$

(ii) f is one-one.

(iii) f is onto.

In otherwords, a bijective homomorphism is said to be an isomorphism.

COSETS :

(1) Left coset of H in G :

Let $(H, *)$ be a subgroup of $(G, *)$

For any $a \in G$, the left coset of H, denoted by $a * H$, is the set, $a * H = \{a * h : h \in H\}$, for all $a \in G$.

(2) Right Coset of H in G :

The right coset of H, denoted by $H * a$ is the set

$$H * a = \{h * a : h \in H\}, \text{ for all } a \in G.$$

INDEX OF H:

The number of distinct left (or right) cosets of H in G is called the index of H in G.

It is denoted by $I_G(H) = \frac{O(G)}{O(H)}$

NOTE: If $a \in H * b$ then $H * a = H * b$ & if $a \in b * H$ then $a * H = b * H$

CAYLEY'S THEOREM :

Every finite group of order 'n' is isomorphic to permutation group of degree 'n'.

Soln: We shall prove this theorem in 3 steps.

Step 1 : Find a set G' of permutation.

Step 2 : Prove G' is a group.

Step 3 : Exhibit an isomorphism $\phi : G \to G'$.

Step 1: Let G be finite group of order 'n'.

Let $a \in G$

Define $f_a : G \to G$ by $f_a(x) = ax$

Since $f_a(x) = f_a(y) \Rightarrow ax = ay$

$\Rightarrow x = y$

$\therefore f_a$ is 1-1

Since, if $y \in G$, then $f_a(a^{-1}y) = aa^{-1}y = y$

$\therefore f_a$ is onto.

Thus f_a is a bijection.

Since G has 'n' elements, f_a is just permutation on 'n' symbols.

Let $G' = \{ f_a / a \in G \}$

Step 2: G' is a group.

Let $f_a, f_b \in G'$

$f_a \circ f_b(x) = f_a(f_b(x)) = f_a(bx) = abx = f_{ab}(x)$

Hence $f_a \circ f_b = f_{ab}$.

Hence G' is closed.

$f_e = G'$ is the identity element.

The inverse of f_a in G' is $f_{a^{-1}}$.

$\therefore G'$ is a group.

Step 3: To prove: G and G' are Isomorphic.

Define $\phi : G \to G'$ by $\phi(a) = f_a$

$$\phi(a) = \phi(b) \Rightarrow f_a = f_b$$
$$\Rightarrow f_a(x) = f_b(x)$$
$$\Rightarrow ax = bx$$
$$\Rightarrow a = b$$

Hence ϕ is 1-1.

Since f_a is onto, ϕ is onto.

Also $\phi(ab) = f_{ab} = f_a \circ f_b = \phi(a) \circ \phi(b)$

$\therefore \phi : G \to G'$ is an Isomorphism.

$\therefore G \simeq G'$.

Hence the proof.