COSETS :

(1) Left coset of H in G :

Let $(H, *)$ be a subgroup of $(G, *)$

For any $a \in G$, the left coset of H, denoted by $a * H$, is the set, $a * H = \{a * h : h \in H\}$, for all $a \in G$.

(2) Right Coset of H in G :

The right coset of H, denoted by $H * a$ is the set

$$H * a = \{h * a : h \in H\}, \text{ for all } a \in G.$$

INDEX OF H :

The number of distinct left (or right) cosets of H in G is called the index of H in G.

It is denoted by $I_G(H) = \frac{O(G)}{O(H)}$

NOTE: If $a \in H * b$ then $H * a = H * b$ & if $a \in b * H$ then $a * H = b * H$

Example:

(1) Let $G = \{1, -1, i, -i\}$

& $H = \{1, -1\}$ is a subgroup of G

The various cosets of H in G are :

$$1H = \{1, -1\} = H = H1$$
$$-1H = \{1, -1\} = H = H(-1)$$

$iH = \{i, -i\} = Hi$

$-iH = \{i, -i\} = iH = H(-i)$

There are two left cosets H, iH &
similarly there are two right cosets H, Hi

(2) Let $G = \{1, a, a^2, a^3\}$ $(a^4 = 1)$ be a group and $H = \{1, a^2\}$ is a subgroup of G under multiplication. Find all the cosets of H & Index of H.

Soln: The right cosets of H in G,

$H \cdot 1 = H * 1 = \{1, a^2\} = H$

$H \cdot a = H * a = \{a, a^3\}$

$H \cdot a^2 = H * a^2 = \{a^2, a^4\} = \{a^2, 1\} = H$

$H \cdot a^3 = H * a^3 = \{a^3, a^5\} = \{a^3, a\} = H * a$

$\therefore$ $H * 1 = H = H * a^2 = \{1, a^2\}$

$H * a = H * a^3 = \{a, a^3\}$ are two distinct r. cosets

Here $G = \{1, a, a^2, a^3\}$ & $H = \{1, a^2\}$

$\Rightarrow O(G) = 4$ & $O(H) = 2$.

$\therefore$ WKT $I_G(H) = \dfrac{O(G)}{O(H)} = \dfrac{4}{2} = 2$.

The no. of distinct right cosets of H in G is 2

(ie) $H * 1 = H * a^2 = \{1, a^2\}$

$H * a = H * a^3 = \{a, a^3\}$

THEOREM:

Any two right (or left) cosets of H in G are either disjoint or identical.

Proof:

Let $H*a$ and $H*b$ be two right cosets of a subgroup H of G.

Let $a, b \in G$

we have to prove that $(H*a) \cap (H*b) = \phi$

(or) $H*a = H*b$

Suppose, $(H*a) \cap (H*b) \neq \phi$

$\exists$ an element $x \in (H*a) \cap (H*b)$

$\Rightarrow x \in H*a$ & $x \in H*b$

Now $x \in H*a$

$\Rightarrow H*x = H*a$ — (1) [since $a \in H*b$ then $H*a = H*b$]

& $x \in H*b$

$\Rightarrow H*x = H*b$ — (2)

From (1) & (2), we have,

$H*x = H*a = H*b = H*x$

$\Rightarrow H*a = H*b$

(or) $(H*a) \cap (H*b) = \phi$