Type II : Symmetric matrix with repeated roots :

① Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Soln :

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The char. eqn is,

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$$

$$c_1 = 0$$

$$c_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -1 - 1 - 1 = -3$$

$$c_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$$

$$\lambda^3 - 3\lambda - 2 = 0 \rightarrow (1)$$

$$\boxed{\lambda = -1, -1, 2}$$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

Case (i) : $\lambda = 2$

$$② \Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cccc} -2 & 1 & 1 & -2 \\ 1 & -2 & 1 & 1 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_2}{3}$$

$\therefore$ The eigen vector is $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Case (ii) : $\lambda = -1$

$$② \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

All the three rows are equal.

Taking the first row,

$$x_1 + x_2 + x_3 = 0$$

Put $x_1 = 0$,

$$x_2 + x_3 = 0$$

$$\Rightarrow x_2 = -x_3$$

$$\Rightarrow \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

The given matrix is a symmetric matrix.

In this matrix, X_3 is orthogonal to X_1 and X_2.

$$\text{Let } X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

X_3 is orthogonal to $X_1 \Rightarrow X_3^T X_1 = 0$

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$a + b + c = 0 \longrightarrow (i)$$

X_3 is orthogonal to $X_2 \Rightarrow X_3^T X_2 = 0$

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$0a + b - c = 0 \longrightarrow (ii)$$

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 1 & +1 \\ 1 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Eigen values : λ : $\quad -1 \quad -1 \quad 2$

Eigen vectors : X : $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(2) Find the eigen values and eigen vectors of

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

Soln: $\lambda = 14, 0, 0$

$$X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} \begin{pmatrix} 13 \\ -2 \\ -3 \end{pmatrix}$$

(3) $$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Soln:

$\lambda = 1, 3, 3$

$$X = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Type 3: Non-Symmetric matrix with repeated roots :

(1) Find all the eigen values and eigen vectors of the matrix $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

Soln:

$$\text{Let } A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

(4)

$C_1 = -1$

$$C_2 = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$= -21$

$$C_3 = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = 45$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0.$$

$$\boxed{\lambda = -3, -3, 5}$$

$(A - \lambda I)X = 0.$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case (i): $\lambda = 5$

$$\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -3 & -7 \\ -6 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Case (ii) : $\lambda = -3$

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

All the three equations are same.

$$x_1 + 2x_2 - 3x_3 = 0$$

Put $x_1 = 0$,

$$2x_2 = 3x_3$$

$$\frac{x_2}{3} = \frac{x_3}{2}$$

$$\therefore \quad X_2 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

Put $x_2 = 0$,

$$x_1 = 3x_3$$

$$\frac{x_1}{3} = \frac{x_3}{1}$$

$$\therefore \quad X_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

Eigen values : $5 \quad -3 \quad -3$

Eigen vectors : $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

(2) $\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$ Soln : $\lambda : -2 \quad 2 \quad 2$

$X : \begin{pmatrix} 4 \\ 1 \\ -7 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(3) $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ Soln : $\lambda : 1 \quad 1 \quad 5$

$X : \begin{pmatrix} 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$