UNIT II

ORTHOGONAL TRANSFORMATION OF A REAL SYMMETRIC MATRIX

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Diagonalization of a real Symmetric matrix:

Transforming a real symmetric matrix A into D by means of the transformation $\boxed{N^T A N = D}$ is known as orthogonal transformation. Here D is the diagonal matrix and N is the matrix whose columns are the normalized eigen vectors of A.

Problems:

(1) Diagonalize the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ by means of an orthogonal transformation ?

Soln:

$$\text{Let } A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Step 1: To find the characteristic equation:

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0 \longrightarrow (1)$$

$$c_1 = 8 + 7 + 3$$

$$= 18$$

$$C_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= (21-16) + (24-4) + (56-36)$$

$$= 45$$

$$C_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 0.$$

Subs C_1, C_2, C_3 in ①,

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0.$$

Step 2: To find the eigen values:

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda = 0, \quad \lambda^2 - 18\lambda + 45 = 0$$

$$\lambda = 3, 15$$

$$\lambda = 0, 3, 15$$

Step 3: To find the eigen vectors:

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow ②$$

Case (i) : $\lambda = 0$

$$(2) \Rightarrow \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking first two rows,

$$\begin{array}{cccccccc} 8 & & -6 & & 2 & & 8 \\ & \times & & \times & & \times & \\ -6 & & 7 & & -4 & & -6 \\ & x_3 & & x_1 & & x_2 & \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 8 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{\cancel{10}\;1} = \frac{x_2}{\cancel{20}\;2} = \frac{x_3}{\cancel{20}\;2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case (ii) : $\lambda = 3$

$$(2) \Rightarrow \begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking first two rows,

$$\begin{array}{cccccccc} 5 & & -6 & & 2 & & 5 \\ & \times & & \times & & \times & \\ -6 & & 4 & & -4 & & -6 \\ & x_3 & & x_1 & & x_2 & \end{array}$$

$$\frac{x_1}{\begin{vmatrix}-6 & 2\\ 4 & -4\end{vmatrix}} = \frac{x_2}{\begin{vmatrix}2 & 5\\ -4 & -6\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}5 & -6\\ -6 & 4\end{vmatrix}}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore$ The eigen vector is $X_2 = \begin{pmatrix}2\\1\\-2\end{pmatrix}$.

<u>case (iii):</u> $\lambda = 15$

$$\begin{pmatrix}-7 & -6 & 2\\ -6 & -8 & -4\\ 2 & -4 & -12\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Taking first 2 rows,

$$\begin{matrix}-7 & -6 & 2 & -7\\ -6 & -8 & -4 & -6\end{matrix}$$

$$\frac{x_1}{\begin{vmatrix}-6 & 2\\ -8 & -4\end{vmatrix}} = \frac{x_2}{\begin{vmatrix}2 & -7\\ -4 & -6\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-7 & -6\\ -6 & -8\end{vmatrix}}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$\therefore$ The eigen vector is $X_3 = \begin{pmatrix}2\\-2\\1\end{pmatrix}$

Hence the modal matrix,

$$M = \begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}$$

Step 4: To find the normalised matrix N:

Normalising each column vector of M.

Dividing each element of first column by 3, second column by 3 & third column by 3, we get the normalized matrix N.

$$N = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

Step 5: Calculate $N^T AN$:

$$N^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$N^T AN = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$N^T AN = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} = D$$

The diagonal elements are the eigen values of A.