UNIT II

ORTHOGONAL TRANSFORMATION OF A REAL SYMMETRIC MATRIX

Quadratic Form :

A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

Ex : $x_1^2 + 5x_1x_2 + 2x_2^2$ is a quadratic form in the variables x_1 and x_2.

Note : Matrix of the quadratic form

$$A = \begin{bmatrix} \text{Coef of } x_1^2 & \frac{1}{2}\text{ coef of } x_1x_2 & \frac{1}{2}\text{ coef of } x_1x_3 \\ \frac{1}{2}\text{ coef of } x_1x_2 & \text{coef of } x_2^2 & \frac{1}{2}\text{ coef of } x_2x_3 \\ \frac{1}{2}\text{ coef of } x_1x_3 & \frac{1}{2}\text{ coef of } x_2x_3 & \text{Coef of } x_3^2 \end{bmatrix}$$

Nature of quadratic form :

Let $Q = X^T A X$ be the given real quadratic form, where A is the matrix of the quadratic form.

Canonical form :

Of a real quadratic form $Q = X^T A X$, the canonical form is $Y^T D Y$ (or) $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$ which is obtained by an orthogonal transformation.

Rank :

If the rank of A is r, then the canonical form of Q consists only 'r' square terms.

Index :

The number of positive square terms in the canonical form is called the index of the quadratic form. It is denoted by s.

Signature :

The difference between the number of positive and negative square terms in the canonical form.

Positive Definite :

If all the eigen values of A are positive.

Negative Definite :

If all the eigen values of A are negative.

Positive Semi Definite :

If atleast one eigen value is zero and the remaining are positive.

Negative Semi Definite :

If atleast one eigen value is zero and the remaining are negative.

InDefinite :

If Some eigen values are positive and some eigen values are negative.

Problems :

(1) Find the matrix of the quadratic form $2x^2 + 3y^2 + 2z^2 + 2xy$

Soln:

$$Q = \begin{bmatrix} \text{Coef of } x^2 & \frac{1}{2}\text{ coef of } xy & \frac{1}{2}\text{ coef of } xz \\ \frac{1}{2}\text{ Coef of } yx & \text{Coef of } y^2 & \frac{1}{2}\text{ coef of } yz \\ \frac{1}{2}\text{ coef of } zx & \frac{1}{2}\text{ coef of } zy & \text{Coef of } z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(2) Write the quadratic form for the following matrix:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Soln:

General form :

$$Q = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1$$

$$Q = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_3x_1$$