Eigen value problems arising from population models : Leslie model

What is the Leslie matrix ?

- Method for representing dynamics of age or Size Structured populations.
- Combines population processes (births and deaths) into a single model
- Generally applied to populations with annual breeding cycle.
- By convention, we use only female part of population.

Problems :

(1) The Leslie model describes age - specified Population growth, as follows. Let the oldest age attained by the females in Some animal population be 6 years. Divide the Population into three age classes of 2 years each. Let the Leslie matrix be

$$L = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} = l_{jk}$$

(i) What is the number of females in each class after 2, 4, 6 years if each class initially consists of 500 females?

(ii) For what initial distribution will the number of females in each class change by the same proportion? What is this rate of change?

Soln:

(i) Given: $L = \begin{pmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix}$

Let $X_0 = \begin{pmatrix} 500 \\ 500 \\ 500 \end{pmatrix}$

After 2 years, the number of females in each class is given by,

$$X_2 = L X_0 = \begin{pmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix} \begin{pmatrix} 500 \\ 500 \\ 500 \end{pmatrix} = \begin{pmatrix} 1350 \\ 300 \\ 150 \end{pmatrix}$$

After 4 years, the number of females in each class is given by,

$$X_4 = L X_2 = \begin{pmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix} \begin{pmatrix} 1350 \\ 300 \\ 150 \end{pmatrix} = \begin{pmatrix} 750 \\ 810 \\ 90 \end{pmatrix}$$

After 6 years, the number of females in each class is given by,

$$X_6 = LX_4 = \begin{pmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix} \begin{pmatrix} 750 \\ 810 \\ 90 \end{pmatrix} = \begin{pmatrix} 1899 \\ 450 \\ 243 \end{pmatrix}$$

(ii) $$L = \begin{pmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix}$$

The characteristic equation is,

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0 \rightarrow (1)$$

$$c_1 = 0 + 0 + 0 = 0 \Rightarrow \boxed{c_1 = 0}$$

$$c_2 = \begin{vmatrix} 0 & 0 \\ 0.3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0.4 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2.3 \\ 0.6 & 0 \end{vmatrix}$$

$$= 0 + 0 + (-1.38)$$

$$\boxed{c_2 = -1.38}$$

$$c_3 = \begin{vmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{vmatrix}$$

$$= 0(0) - 2.3(0) + 0.4(0.18)$$

$$\boxed{c_3 = 0.072}$$

Subs c_1, c_2, c_3 in (1),

$$\lambda^3 - 0\lambda^2 - 1.38\lambda - 0.072 = 0$$

$$\lambda^3 - 1.38\lambda - 0.072 = 0.$$

Solving, we get,

$$\lambda = 1.2, -1.14, -0.05$$

$\therefore$ Let $\lambda = 1.2$ (Take only positive root for growth)

To find eigen vector:

$$(A - \lambda I)X = 0$$

$$(L - \lambda I)X = 0. \quad \text{Here } A = L$$

$$\begin{pmatrix} 0-\lambda & 2.3 & 0.4 \\ 0.6 & 0-\lambda & 0 \\ 0 & 0.3 & 0-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

$\lambda = 1.2$

$$(2) \Rightarrow \begin{pmatrix} -1.2 & 2.3 & 0.4 \\ 0.6 & -1.2 & 0 \\ 0 & 0.3 & -1.2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking first and Second row,

$$\begin{array}{cccc} -1.2 & 2.3 & 0.4 & -1.2 \\ 0.6 & -1.2 & 0 & 0.6 \\ x_3 & x_1 & x_2 & \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 2.3 & 0.4 \\ -1.2 & 0 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0.4 & -1.2 \\ 0 & 0.6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1.2 & 2.3 \\ 0.6 & -1.2 \end{vmatrix}}$$

$$\frac{x_1}{0.48} = \frac{x_2}{0.24} = \frac{x_3}{0.06}$$

$$\frac{x_1}{8} = \frac{x_2}{4} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$$

$$\therefore \; 8x + 4x + 1x = 1500$$

$$13x = 1500$$

$$x = \frac{1500}{13} = 115.4$$

$$\boxed{x = 115.4}$$

In class I

$$8x = 8(115.4) = 923$$

In class II

$$4x = 4(115.4) = 462$$

In class III

$$1x = 1(115.4) \simeq 115.$$

The growth rate will be $\boxed{\lambda = 1.2}$ per 3 years.