SNS COLLEGE OF TECHNOLOGY

(AN AUTONOMOUS INSTITUTION)

COIMBATORE-35

DEPARTMENT OF MATHEMATICS

UNIT I (Two mark)

1. Find the Characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Solution:- Let A=$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

The Characteristic equation is $\lambda^2 - s_1\lambda + s_2 = 0$

$s_1 =$ **Sum of the main diagonal elements** $= 1 + 2 = 3$

$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

Hence the required Characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

2. Find the Characteristic polynomial of $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Solution:- The Characteristic polynomial is $|A - \lambda I| = \lambda^2 - s_1\lambda + s_2$

$s_1 =$ **Sum of the main diagonal elements** $= 1 + 3 = 4$

$s_2 = |A| = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3 - 8 = -5$

Hence the required Characteristic polynomial is $\lambda^2 - 4\lambda - 5$

3. The product of two Eigen values of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ ***is*** 16**. Find the third Eigen value.**

Solution:-Let the Eigen values of the matrix A ***be*** $\lambda_1, \lambda_2, \lambda_3$

Given $\lambda_1\lambda_2 = 16$

We know that $\lambda_1\lambda_2\lambda_3 = |A|$**, [Since the product of the Eigen values is equal to the determinant of the matrix]**

$$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 32$$

$$16\lambda_3 = 32 \quad [\because \lambda_1\lambda_2 = 16]$$

$$\lambda_3 = \frac{32}{16} = 2$$

4. **The Eigen value of the matrix**

$$\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$ **are 0 and 1. Find the other Eigen value.**

Solution:- WKT Sum of the Eigen values = Sum of the main diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 11 + (-2) + (-6)$$

$$0 + 1 + \lambda_3 = 3$$

$$\therefore \lambda_3 = 2$$

Therefore the third Eigen value is 2.

5. **Find the Sum and Product of the Eigen values of the matrix** $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solution:- WKT Sum of the Eigen values = Sum of the main diagonal elements

=2+2+2=6

Product of the Eigen values = $|A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 8 - 2 = 6$

6. **One of the Eigen values of the matrix** $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$ **is -9 Find the other two Eigen values**

Solution:- WKT Sum of the Eigen values = Sum of the main diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 7 + (-8) + (-8)$$

$$\lambda_1 + \lambda_2 - 9 = -9$$

$$\lambda_1 + \lambda_2 = 0 \ldots\ldots\ldots\ldots (1)$$

Product of the Eigen values = $|A|$

7. **The matrix** $A=\begin{bmatrix}1 & 2 & 1\\2 & 0 & -2\\1 & 2 & 3\end{bmatrix}$ **is singular.One of the Eigen Value is 2. Find the other two Eigen Values.**

Solution:-- **Sum of the Eigen Values** $=\lambda_1+\lambda_2+\lambda_3$

Sum of the main diagonal elements of A = 1+0+3

WKT, Sum of the Eigen Values=Sum of the main diagonal elements

$$\lambda_1+\lambda_2+\lambda_3=1+0+3$$
$$2+\lambda_2+\lambda_3=4$$
$$\lambda_2+\lambda_3=2$$

Product of the Eigen Values $=|A|$

$$\lambda_1\lambda_2\lambda_3=|A|=\begin{vmatrix}1 & 2 & 1\\2 & 0 & -2\\1 & 2 & 3\end{vmatrix}$$
$$2\lambda_2\lambda_3=-8$$
$$\lambda_2\lambda_3=-4$$

WKT, x^2- **(Sum of the Eigen Value)** $x+$ **Product of the Eigen values**

$$x^2-2x+(-4)=0$$
$$x^2-2x-4=0$$
$$x=\frac{2\pm\sqrt{20}}{2}=1\pm\sqrt{5}$$
$$\therefore\quad \lambda_2=1+\sqrt{5}\quad ,\lambda_3=1-\sqrt{5}$$

8. **Find the Characteristic equation of the matrix** $\begin{pmatrix}1 & 2\\0 & 2\end{pmatrix}$ **and get the Eigen Values.**

Solution:-- Given matrix is a triangular matrix. Hence the Eigen Values are 1,2.

The characteristic of the given matrix is,

$$\lambda^2-(Sum\ of\ the\ Eigen\ value-S_1)\lambda+\Pr oduct\ of\ the\ Eigen\ value-S_2=0$$
$$\Rightarrow\lambda^2-(1+2)\lambda+(1)(2)=0$$
$$\Rightarrow\lambda^2-3\lambda+2=0$$

9. **If** $\alpha\ and\ \beta$ **are the Eigen Values of A=** $\begin{bmatrix}3 & -1\\-1 & 5\end{bmatrix}$ **form the matrix whose Eigen Values are** $\alpha^3\ and\ \beta^3$

Solution:-- WKT "If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots, \lambda_n$ are the Eigen Values of a matrix A, then A^m has Eigen Values $\lambda^m{}_1, \lambda_2{}^m, \lambda_3{}^m, \ldots\ldots\ldots, \lambda_n{}^m$ (m being positive integer)"

Let $A = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$ α, β **be the Eigen Values**

Now,

$$A^2 = AA = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 38 & -50 \\ -50 & 138 \end{bmatrix}$$

Hence $A^3 = \begin{bmatrix} 38 & -50 \\ -50 & 138 \end{bmatrix}$ **is the matrix whose Eigen Values be** α^3 **&** β^3

10. If 1,1,5 are the Eigen Values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$**, find the Eigen Values of 5A.**

Solution:-- WKT " If $\lambda_1, \lambda_2, \lambda_3$ be the Eigen Values of A, then $k\lambda_1, k\lambda_2, k\lambda_3$ be the Eigen Values of kA".

$\therefore$ **The Eigen Values of 5A are 5,5,25.**

11. If 2,3 are the Eigen Values of $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$**, find the Eigen Values of a.**

Solution:-- Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen Values of A.

WKT, Sum of the Eigen Values=Sum of the main diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 + 2$$

$$2 + 3 + \lambda_3 = 6$$

$$\lambda_3 = 1$$

Product of the Eigen Values $= |A|$

$$\lambda_1 \lambda_2 \lambda_3 = |A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix}$$

$$2 \times 3 \times 1 = 8 - 2a$$

$$6 - 8 = -2a$$

$$\therefore a = 1$$

12. **Two Eigen values of** $A=\begin{bmatrix}4 & 6 & 6\\ 1 & 2 & 2\\ -1 & -5 & -2\end{bmatrix}$ **are equal and they are double the third.Find the Eigen values of** A^2

Solution:-- Let the third Eigen Value be λ

The remaining Eigen Values are 2 λ,2 λ

WKT, Sum of the Eigen Values = Sum of the main diagonal elements

$$2\lambda+2\lambda+\lambda=4+3+(-2)$$
$$5\lambda=5$$
$$\lambda=1$$

$\therefore$**Eigen Values of A are 2,2,1.**

Hence Eigen Values of A^2 **are** $2^2,2^2,1^2$ **(i.e) 4,4,1**

13. **Prove that Eigen Values of** $-3A^{-1}$ **are the Values of same as those** $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$

Solution:-- The Characteristic equation of A is

$$\lambda^2-s_1\lambda+s_2=0$$

s_1 = **Sum of the main diagonal elements** $=1+1=2$

$$s_2=|A|=\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}=1-4=-3$$

The Characteristic equation is

$$\lambda^2-2\lambda-3=0$$
$$(\lambda-3)(\lambda+1)=0$$
$$\Rightarrow\lambda=3,-1$$

Hence the Eigen values of A are -1,3.

Hence the Eigen values of A^{-1} **are -1,1/3**

Then the Eigen Values of $-3A^{-1}$ ***are*** $-3(-1),-3\left(\frac{1}{3}\right)$

$$ie.,,3,-1$$

Hence Eigen Values of A and $-3A^{-1}$ **are same.**

14. **Sum of squares of the Eigen Values of** $A=\begin{bmatrix}1 & 7 & 5\\ 0 & 2 & 9\\ 0 & 0 & 5\end{bmatrix}$ **is?**

Solution:--The Characteristic equation of the given matrix is $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & 7 & 5 \\ 0 & 2-\lambda & 9 \\ 0 & 0 & 5-\lambda \end{vmatrix}=0$$

$$(1-\lambda)(2-\lambda)(5-\lambda)=0$$

$$\therefore \lambda = 1,2,5$$

Sum of squares of Eigen Values = $1^2+2^2+5^2=30$

Define Quadratic Form.

15. A Homogeneous polynomial of second degree in any number of variables is called a quadratic form.

Write the matrix of the quadratic form

16. $2x^2+8z^2+4xy-10xz-2yz$

Solution:

$$Q=\begin{bmatrix} Coeff\ x^2 & \frac{1}{2}Coeff\ xy & \frac{1}{2}Coeff\ xz \\ \frac{1}{2}Coeff\ xy & Coeff\ y^2 & \frac{1}{2}Coeff\ yz \\ \frac{1}{2}Coeff\ xz & \frac{1}{2}Coeff\ yz & Coeff\ z^2 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$$

17. Determine the nature of the Quadratic form $f(x_1,x_2,x_3)=x_1^2+2x_2^2$

Solution:--The matrix of the Quadratic form is

$$Q=\begin{bmatrix} Coeff\ x_1^2 & \frac{1}{2}Coeff\ x_1x_2 & \frac{1}{2}Coeff\ x_1x_3 \\ \frac{1}{2}Coeff\ x_2x_1 & Coeff\ x_2^2 & \frac{1}{2}Coeff\ x_2x_3 \\ \frac{1}{2}Coeff\ x_3x_1 & \frac{1}{2}Coeff\ x_3x_2 & Coeff\ x_3^2 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_1 = |1| = 1 \quad (+ve)$$

$$D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 \quad (+ve)$$

$$D_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\therefore$ The Quadratic form is said to be positive Semi-definite.

18. If the Sum of the Eigen Values of the matrix of the quadratic form equal to zero, then what will be the nature of the quadratic form.

Solution:--Given $\lambda_1 + \lambda_2 + \lambda_3 = 0$

Case (i) All +ve is not possible.

Case(ii) All –ve is not possible.

Case (iii) both positive and negative is possible.

$\therefore$ Nature of the Quadratic form is indefinite.

19. Find the Eigen Vector corresponding to the Eigen Value '1' of matrix $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

Solution:-- Given $\lambda = 1$

The Characteristic equation is $(A - \lambda I)x = 0$

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\sin ce \quad \lambda = 1$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

Three equations are equal. So we give arbitrary values to x_1 & x_2.

Let $x_1 = 1, x_2 = -1$

$$x_1 + 2x_2 + x_3 = 0$$

$$\Rightarrow 1 - 2 + x_3 = 0$$

$$\Rightarrow x_3 = 1$$

$\therefore$ The Eigen Vector $X = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

20. Find the Eigen Value of $\begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix}$ corresponding to the Eigen Vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution:--WKT,

$$AX = \lambda X$$

$$Here \quad A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$AX = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda x$$

$$\lambda = 2$$

21. If A is an orthogonal matrix , Show that A^{-1} is also orthogonal.

Solution:-- Since A is orthogonal matrix

ie., $A^T = A^{-1}$ ....(1)

To prove: A^{-1} is orthogonal. ie.,

$$\left(A^{-1}\right)^T = \left(A^{-1}\right)^{-1}$$

$$Let \quad B = A^{-1} \Rightarrow B^T = B^{-1}$$

$$B^T = \left(A^{-1}\right)^T = \left(A^{-1}\right)^T = \left(A^{-1}\right)^{-1}$$

$$B^T = B^{-1}$$

$$\left(A^{-1}\right)^T = \left(A^{-1}\right)^{-1}$$

$\therefore$ A^{-1} is orthogonal matrix.

22. Can $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be diagonalized? why?

Solution:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ can be diagonalized ,since it is a symmetric and non singular matrix.

23. State Cayley – Hamilton theorem.

Solution : Every square matrix satisfies its own characteristic equation.

24. Find the Eigen value of Adj A ,if $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution :

Since A is an Triangular matrix,The eigen values of A are 3,4,1.

$|A| = 3x4x1=12$

$$\frac{\text{adj}\,A}{|A|} = A^{-1} \Rightarrow \text{adj}\,A = A^{-1}\,|A|$$

$=12A^{-1}$

The Eigen value of A^{-1} are $\frac{1}{3}, \frac{1}{4}, 1$

The Eigen value of adj A are $12\times\frac{1}{3}, 12\times\frac{1}{4}, 12\times 1$

$\therefore$ 4,3,12

25. Show that $A=\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$ is orthogonal.

Solution:

$$\text{Let } A^T = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= I = AA^T$

Hence A is orthogonal.