3. verify cayley hamilton theorem for $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$, find $A^{-1}$

Soln.

The characteristic eqn is, $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

$D_1 = 1+1-2 = 0$

$D_2 = (-2-0) + (-2+1) + (1-0) = -2-1+1$
$= -2$

$D_3 = 1(-2) - 0 - 1(-1) = -2+1 = -1$

$\therefore \lambda^3 - 2\lambda + 1 = 0$

To prove: $A^3 - 2A + I = 0$

$$A^2 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 5 & 1 & -3 \\ -1 & 0 & 3 \end{bmatrix}$$

$$A^3 = A \times A^2 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 5 & 1 & -3 \\ -1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 4 & 1 & 2 \\ 2 & 0 & -5 \end{bmatrix}$$

$$\therefore A^3 - 2A + I = \begin{bmatrix} 1 & 0 & -2 \\ 4 & 1 & 2 \\ 2 & 0 & -5 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 0$$

Hence cayley hamilton theorem is verified.

To find $A^{-1}$:

$(\times A^{-1})$ $A^2 - 2I + A^{-1} = 0$

$$A^{-1} = 2I - A^2$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 5 & 1 & -3 \\ -1 & 0 & 3 \end{bmatrix}$$

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$$\therefore A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

4. Use Cayley-Hamilton theorem to find the value of $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I=0$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.

Soln:

The characteristic eqn. is, $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

$D_1 = 2+1+2 = 5$

$D_2 = 2+3+2 = 7$

$D_3 = |A| = 3$

$\therefore \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley-Hamilton theorem, we get

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$\begin{array}{r|l} & A^5 + A \\ \hline A^3-5A^2+7A-3I & A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I \\ & A^8-5A^7+7A^6-3A^5 \\ \hline & A^4-5A^3+8A^2-2A \\ & A^4-5A^3+7A^2-3A \\ \hline & A^2+A+I \end{array}$$

$\therefore A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

$= (A^3-5A^2+7A-3I)(A^5+A) + (A^2+A+I)$

$= 0(A^5+A) + (A^2+A+I)$

$= A^2+A+I \quad = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$

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5). Use Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ to express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A & find value.

Soln.

CE $\lambda^2 - D_1\lambda + D_2 = 0$

$D_1 = 1+3 = 4$

$D_2 = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3-8 = -5$

$\therefore \lambda^2 - 4\lambda - 5 = 0$

~~To prove:~~

By CH theorem, $A^2 - 4A - 5I = 0$

To find the value $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$.

$$\begin{array}{r|l} & A^3 - 2A + 3 \\ \hline A^2-4A-5I & A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I \\ & A^5 - 4A^4 - 5A^3 \\ & \scriptstyle(-)\quad(+)\quad(+) \\ \hline & -2A^3 + 11A^2 - A \\ & -2A^3 + 8A^2 + 10A \\ & \scriptstyle(+)\quad(-)\quad(-) \\ \hline & 3A^2 - 11A - 10I \\ & 3A^2 - 12A - 15I \\ & \scriptstyle(-)\quad(+)\quad(+) \\ \hline & A + 5I \\ \hline \end{array}$$

$\therefore A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$= (A^2 - 4A - 5I)(A^3 - 2A + 3) + (A + 5I)$

$= 0 + (A + 5I)$

$= A + 5I$, is a linear polynomial in A.

And $A + 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$

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