Cayley-Hamilton theorem:
Every square matrix satisfies its own characteristic equation.

Uses of Cayley-Hamilton Theorem:
To calculate (i) the positive integral powers of A
(ii) the inverse of a non-singular square matrix A.

1. Verify Cayley-Hamilton theorem for the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

Soln.

Let $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

The characteristic eqn. is $\lambda^2 - D_1\lambda + D_2 = 0$

$D_1 = 1+1 = 2$

$D_2 = |A| = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1+4$
$= 5$

$\therefore \lambda^2 - 2\lambda + 5 = 0$

To prove: $A^2 - 2A + 5I = 0$

Now, $A^2 = A \cdot A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$

and $A^2 - 2A + 5I = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} - 2\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore$ The given matrix satisfies its own characteristic eqn.

Hence Cayley Hamilton theorem is verified.

2. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find A^4 and A^{-1}.

Soln.

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characteristic eqn. is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

$D_1 = 2+2+2 = 6$

$D_2 =$ Sum of minors of main diagonal elts.

$= (4-1) + (4-2) + (4-1)$

$= 3+2+3$

$= 8$

$D_3 = |A|$

$= 2(4-1) + 1(-2+1) + 2(1-2)$

$= 3$

$\therefore \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove: $A^3 - 6A^2 + 8A - 3I = 0$

Now

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A \times A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

and $A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + 8\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence cayley hamilton theorem is verified.

To find A^{-1}:

$$A^3-6A^2+8A-3I=0$$

$$(\times A^{-1}) \quad A^2-6A+8I-3A^{-1}=0$$

$$3A^{-1}=A^2-6A+8I$$

$$A^{-1}=\frac{1}{3}[A^2-6A+8I]$$

$$A^{-1}=\frac{1}{3}\left\{\begin{bmatrix}7 & -6 & 9\\ -5 & 6 & -6\\ 5 & -5 & 7\end{bmatrix}-6\begin{bmatrix}2 & -1 & 2\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}+8\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right\}$$

$$=\frac{1}{3}\begin{bmatrix}3 & 0 & -3\\ 1 & 2 & 0\\ -1 & 1 & 3\end{bmatrix}$$

To find A^4:

$$(\times A) \quad A^4-6A^3+8A^2-3A=0$$

$$A^4=6A^3-8A^2+3A$$

$$=\cancel{6\,[29 \;\; -28 \;\; 38}$$

$$=6[6A^2-8A+3I]-8A^2+3A$$

$$=36A^2-48A+18I-8A^2+3A$$

$$A^4=28A^2-45A+18I$$

$$=28\begin{bmatrix}7 & -6 & 9\\ -5 & 6 & -6\\ 5 & -5 & 7\end{bmatrix}-45\begin{bmatrix}2 & -1 & 2\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}+18\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A^4=\begin{bmatrix}124 & -123 & 162\\ -95 & 96 & -123\\ 95 & -95 & 124\end{bmatrix}$$