



LAPLACE TRANSFORMS :-

Jouier transform represents continuous time signal in terms of complex sinusoids. (ie) $e^{j\omega t}$

Laplace transform represents continuous time signals in terms of complex exponentials (ie) e^{-st}

continuous time systems can also be analysed effectively using Laplace transforms. Laplace transform of impulse response is called System Function (or)

● Transfer Function.

Types of Laplace transform :-

- 1) Bilateral Laplace transform
- 2) Unilateral Laplace transform.

Laplace Transform :-

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$s = \sigma + j\omega$$

σ = attenuation constant

ω = complex frequency.

Unilateral Laplace Transform :-

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt$$

Inverse Laplace Transform :-

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$



Relationship between Fourier Transform and Laplace Transform

Fourier Transform :-

$$x(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \quad (0\pi)$$

$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \rightarrow \textcircled{1}$$

Laplace Transform :-

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$s = \sigma + j\omega$$

$$x(s) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt \quad \rightarrow \textcircled{2}$$

comparing eqn ① & ②

Laplace Transform of $x(t)$ is basically a Fourier transform of $x(t) e^{-\sigma t}$

$$L[x(t)] = x(s) = F[x(t) e^{-\sigma t}]$$

$\sigma = 0$, becomes

$$L[x(t)] = F[x(t)]$$

$$\downarrow$$

$$x(s) = x(j\omega), \text{ where } s = j\omega$$

$s = j\omega$ indicates Imaginary axis in complex plane.

convergence of Laplace Transform :-

Laplace transform is basically a Fourier transform of $x(t) e^{-\sigma t}$. Fourier transform of $x(t) e^{-\sigma t}$ exist, then $L[x(t)]$ also exist.



Fourier transform of $x(t) e^{-\sigma t}$ must be absolutely integrable.

$$\int_{-\infty}^{\infty} |x(t) e^{-\sigma t}| dt < \infty$$

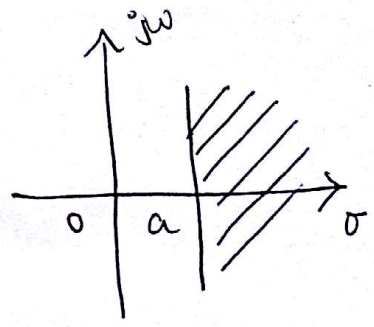
Roc :- [Region of convergence]

The Range of value of σ for which the Laplace transform convergence is called region of convergence.

Problems :-

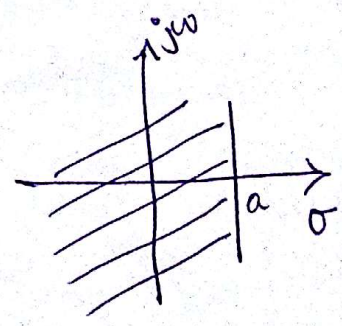
(1) Find the Laplace transform of $x(t) = e^{at} u(t)$ and plot its Roc.

$$\begin{aligned}
 x(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \\
 &= \int_{-\infty}^{\infty} e^{at} u(t) e^{-st} dt \\
 &= \int_0^{\infty} e^{at} e^{-st} dt \\
 &= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} \Rightarrow \frac{1}{s-a}, \quad \begin{matrix} s-a > 0 \\ s > a \end{matrix}
 \end{aligned}$$



(2) Find the Laplace transform of $x(t) = e^{-at} u(-t)$

$$\begin{aligned}
 x(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \\
 &= \int_{-\infty}^{\infty} e^{-at} u(-t) e^{-st} dt \\
 &= \int_{-\infty}^0 -e^{-at} e^{-st} dt \\
 &= \left[\frac{-e^{-t(s-a)}}{-(s-a)} \right]_{-\infty}^0 \Rightarrow \frac{1}{s-a}, \quad \begin{matrix} s-a < 0 \\ s < a \end{matrix}
 \end{aligned}$$



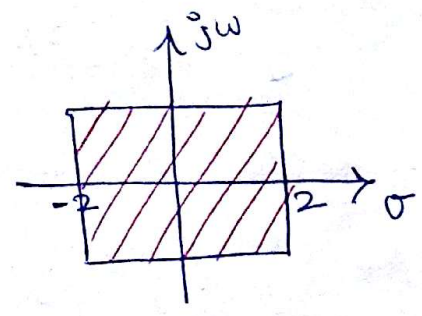


3) $x(t) = e^{-2t} u(t) - e^{2t} u(-t)$

$$x(s) = \frac{1}{s+2} - \frac{1}{s-2}$$

$$s+2 > 0 \quad s-2 > 0$$

$$s > -2 \quad s < 2$$



$$-2 < s < 2$$

4) Find Laplace transform of $r(t)$

$$x(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} t u(t) e^{-st} dt$$

$u = t$
 $u' = 1$
 $dv = e^{-st}$
 $v = \frac{e^{-st}}{-s}$
 $v_1 = \frac{e^{-st}}{s^2}$

$$= \int_0^{\infty} t e^{-st} dt$$

$$= \left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^{\infty} = \frac{1}{s^2}$$

5) Find Laplace Transform of $x(t) = e^{-at} \cos \omega t u(t)$

$$x(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} \cos \omega t u(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2} \right) u(t) e^{-st} dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-at} (e^{j\omega t} + e^{-j\omega t}) u(t) e^{-st} dt$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{-at} e^{j\omega t} u(t) e^{-st} dt + \int_{-\infty}^{\infty} e^{-at} e^{-j\omega t} u(t) e^{-st} dt \right]$$



$$\therefore = \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{(j\omega-a)t} u(t) e^{-st} dt + \int_{-\infty}^{\infty} e^{(j\omega+a)t} u(t) e^{-st} dt \right]$$

$$= \frac{1}{2} \left[L \left[e^{(j\omega-a)t} u(t) \right] + L \left[e^{(j\omega+a)t} u(t) \right] \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-j\omega+a} + \frac{1}{s+j\omega+a} \right]$$

$$= \frac{1}{2} \left(\frac{2s+2a}{s^2+\omega^2+a^2} \right) \Rightarrow \frac{1}{2} \left(\frac{2(s+a)}{s^2+\omega^2+a^2} \right)$$

$$X(s) = \frac{s+a}{s^2+a^2+\omega^2}$$

(HW)

(b)

Find the Laplace Transform of $x(t) = e^{-at} \sin \omega t u(t)$

$$X(s) = \frac{\omega}{(s+a)^2 + \omega^2}$$

(7)

Find the Laplace transform of $x(t) = \sinh at$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \sinh at e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{e^{at} - e^{-at}}{2} \right) e^{-st} dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} (e^{at} - e^{-at}) e^{-st} dt$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{at} e^{-st} dt - \int_{-\infty}^{\infty} e^{-at} e^{-st} dt \right]$$

$$\left[\sinh at = \frac{e^{at} - e^{-at}}{2} \right]$$



$$= \frac{1}{2} \left[L(e^{at}) - L(e^{-at}) \right]$$

$$= \frac{1}{2} \left(\frac{1}{s-a} - \frac{1}{s+a} \right)$$

$$= \frac{1}{2} \left(\frac{s+a-s+a}{s^2-a^2} \right) = \frac{1}{2} \left[\frac{2a}{s^2-a^2} \right] = \frac{a}{s^2+a^2}$$

8) Find the Laplace transform of $x(t) = \cosh at$

$$X(s) = \frac{s}{s^2+a^2}$$

9) Find the Laplace transform of $x(t) = e^{-at} \cosh bt$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} \cosh bt e^{-st} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} \left(\frac{e^{bt} + e^{-bt}}{2} \right) e^{-st} dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-at} (e^{bt} + e^{-bt}) e^{-st} dt$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{-at} e^{bt} e^{-st} dt + \int_{-\infty}^{\infty} e^{-at} e^{-bt} e^{-st} dt \right]$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{(b-a)t} e^{-st} dt + \int_{-\infty}^{\infty} e^{-(b+a)t} e^{-st} dt \right]$$

$$= \frac{1}{2} \left[L[e^{(b-a)t}] + L[e^{-(b+a)t}] \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-b+a} + \frac{1}{s+b+a} \right]$$



$$= \frac{1}{2} \left[\frac{s+b+a+s-b+a}{s^2-b^2+a^2} \right] \Rightarrow \frac{1}{2} \left[\frac{2s+2a}{s^2-b^2+a^2} \right]$$

$$= \frac{1}{2} \left[\frac{2(s+a)}{s^2-b^2+a^2} \right]$$

$$x(s) = \frac{s+a}{(s+a)^2 - b^2}$$

HW
10)

Find the Laplace transform of $x(t) = e^{-at} \sin bt$

$$x(s) = \frac{b}{(s+a)^2 - b^2}$$

Unilateral Laplace Transform :-

$$x(s) = \int_0^{\infty} x(t) e^{-st} dt$$

Roc neednot be specified with Unilateral Laplace transform.

Inverse Laplace transform using partial fraction

Method :-

$$L [e^{at} u(t)] = \frac{1}{s-a}, s > a$$

$$L [-e^{at} u(-t)] = \frac{1}{s-a}, s < a$$

$$L [e^{-at} u(t)] = \frac{1}{s+a}, s > -a$$

$$L [-e^{-at} u(-t)] = \frac{1}{s+a}, s < -a$$