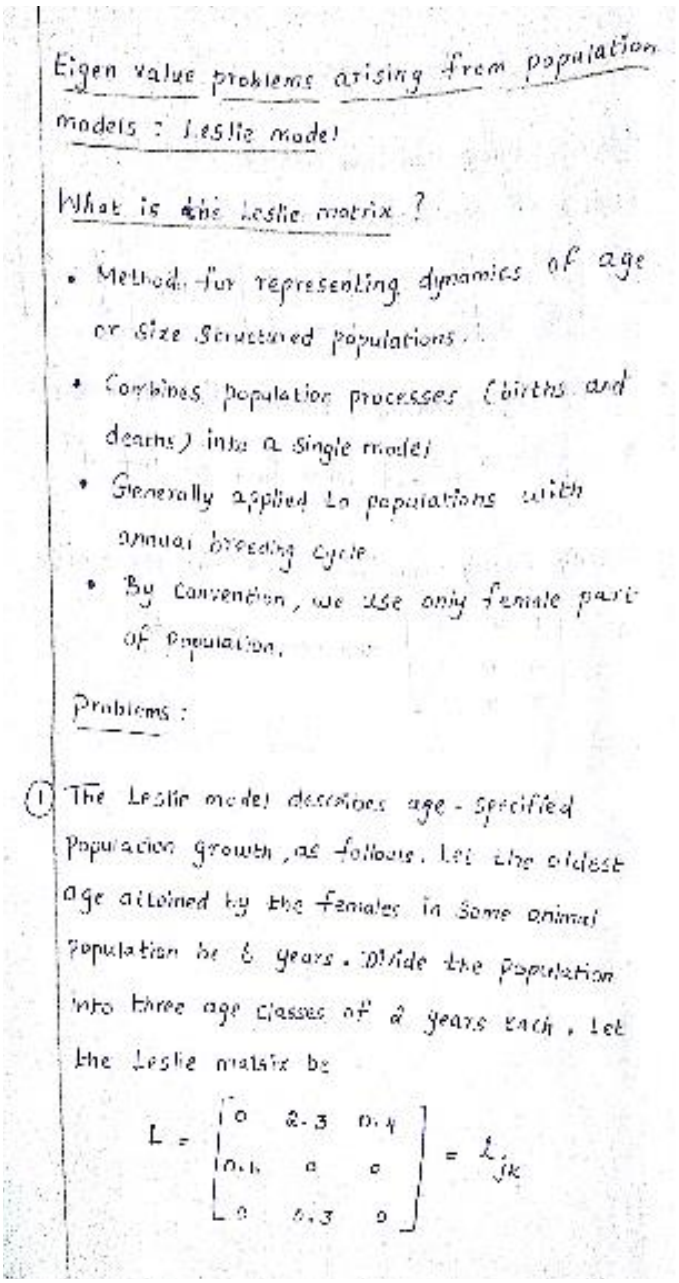
23MAT101 - MATRICES AND CALCULUS

UNIT-I MATRIX EIGEN VALUE PROBLEM



Eigen value problems arising from population models : Leslie model

What is the Leslie matrix ?

- Method for representing dynamics of age or size Structured populations.
- Combines population processes (births and deaths) into a single model
- Generally applied to populations with annual breeding cycle
- By convention, we use only female part of population.

Problems :

(1) The Leslie model describes age-specified Population growth, as follows. Let the oldest age attained by the females in some animal Population be 6 years. Divide the population into three age classes of 2 years each. Let the Leslie matrix be

$$L = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} = L_{jk}$$

(i) What is the number of females in each class after 2, 4, 6 years if each class initially consists of 500 females ?

(ii) For what initial distribution will the number of females in each class change by the same proportion ? What is this rate of change ?

Soln:

(i) Given : $L = \begin{pmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix}$

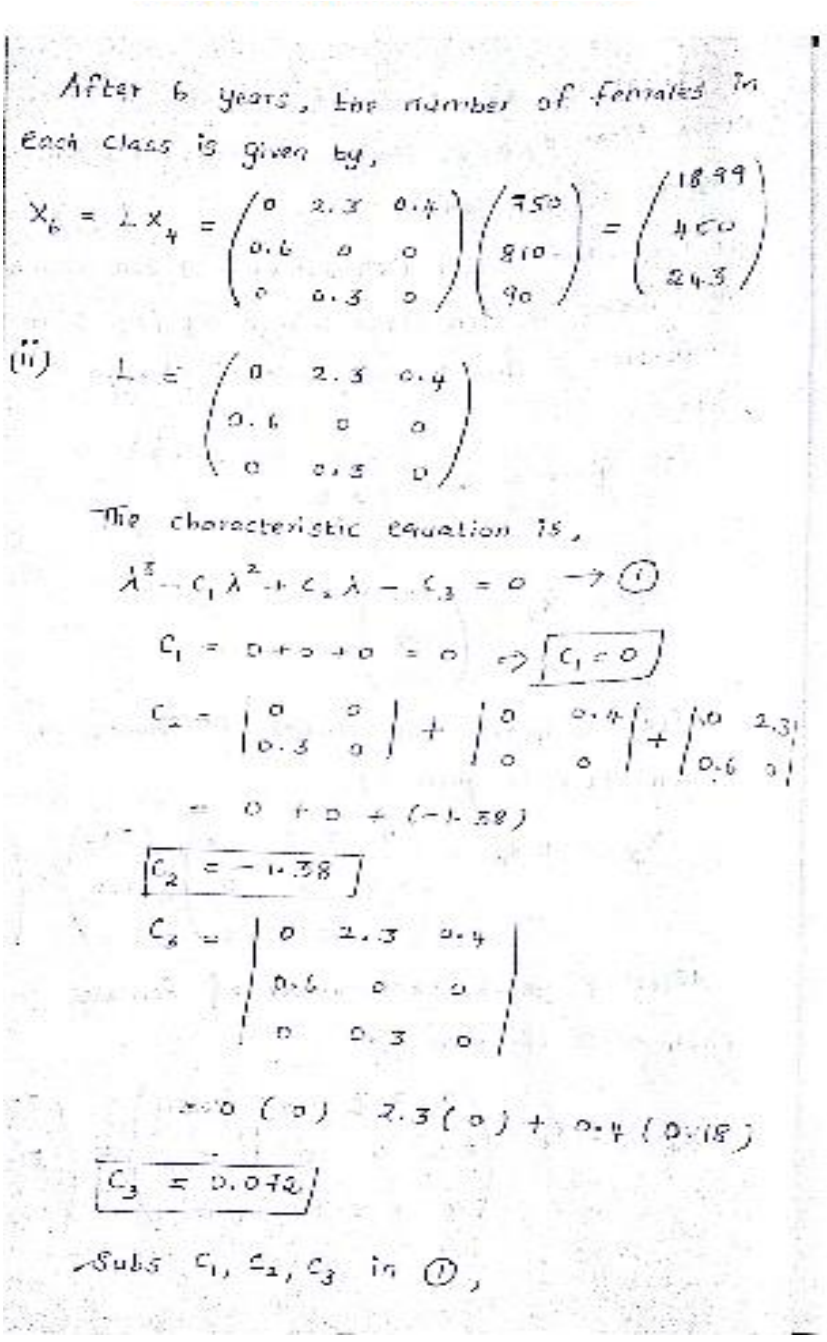
Let $X_0 = \begin{pmatrix} 500 \\ 500 \\ 500 \end{pmatrix}$

After 2 years, the number of females in each class is given by,

$$X_2 = L X_0 = \begin{pmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix} \begin{pmatrix} 500 \\ 500 \\ 500 \end{pmatrix} = \begin{pmatrix} 1350 \\ 300 \\ 150 \end{pmatrix}$$

After 4 years, the number of females in each class is given by,

$$X_4 = L X_2 = \begin{pmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix} \begin{pmatrix} 1350 \\ 300 \\ 150 \end{pmatrix} = \begin{pmatrix} 750 \\ 810 \\ 90 \end{pmatrix}$$



After 6 years, the number of females in each class is given by,

$$X_6 = LX_4 = \begin{pmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix} \begin{pmatrix} 750 \\ 810 \\ 90 \end{pmatrix} = \begin{pmatrix} 1899 \\ 450 \\ 243 \end{pmatrix}$$

(ii) $L = \begin{pmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix}$

The characteristic equation is,

$$\lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0 \rightarrow (1)$$

$$C_1 = 0 + 0 + 0 = 0 \Rightarrow \boxed{C_1 = 0}$$

$$C_2 = \begin{vmatrix} 0 & 0 \\ 0.3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0.4 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2.3 \\ 0.6 & 0 \end{vmatrix}$$

$$= 0 + 0 + (-1.38)$$

$$\boxed{C_2 = -1.38}$$

$$C_3 = \begin{vmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{vmatrix}$$

$$= 0(0) - 2.3(0) + 0.4(0.18)$$

$$\boxed{C_3 = 0.072}$$

Subs C_1, C_2, C_3 in (1),

$\lambda^3 - 0\lambda^2 - 1.38\lambda - 0.072 = 0$

$\lambda^3 - 1.38\lambda - 0.072 = 0$

Solving, we get,

$\lambda = 1.2, -1.14, -0.05$

$\therefore$ Let $\lambda = 1.2$ (Take only positive root for growth)

To find eigen vector

$(A - \lambda I)X = 0$

$(L - \lambda I)X = 0$ Here $A = L$

$$\begin{pmatrix} 0-\lambda & 2.3 & 0.4 \\ 0.6 & 0-\lambda & 0 \\ 0 & 0.3 & 0-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

$\lambda = 1.2$

$$(2) \rightarrow \begin{pmatrix} -1.2 & 2.3 & 0.4 \\ 0.6 & -1.2 & 0 \\ 0 & 0.3 & -1.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking first and second row,

$$\begin{matrix} -1.2 & 2.3 & 0.4 & -1.2 \\ 0.6 & -1.2 & 0 & 0.6 \\ x_3 & \lambda & x_2 & \end{matrix}$$

$$\frac{x_1}{\begin{vmatrix} 2.3 & 0.4 \\ 1.2 & 0 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0.4 & -1.2 \\ 0 & 0.6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1.2 & 2.3 \\ 0.6 & -1.2 \end{vmatrix}}$$

$$\frac{x_1}{0.48} = \frac{x_2}{0.24} = \frac{x_3}{0.06}$$

$$\frac{x_1}{8} = \frac{x_2}{4} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$$

$\therefore$ $8x + 4x + 1x = 1500$

$$13x = 1500$$

$$x = \frac{1500}{13} = 115.4$$

$$\boxed{x = 115.4}$$

In class I

$$8x = 8(115.4) = 923$$

In class II

$$4x = 4(115.4) = 462.$$

In class III

$$1x = 1(115.4) = 115.$$

The growth rate will be $\boxed{\lambda = 1.2}$ per 3 years.