



## DEPARTMENT OF MATHEMATICS

### 23MAT101 - MATRICES AND CALCULUS

#### UNIT-II ORTHOGONAL TRANSFORMATION OF A REAL SYMMETRIC MATRIX

#### UNIT - II

#### ORTHOGONAL TRANSFORMATION OF

#### REAL SYMMETRIC MATRIX

#### Diagonalization of a real symmetric matrix:

Transforming a real symmetric matrix  $A$  into  $D$  by means of the transformation  $N^T A N = D$  is known as orthogonal transformation. Here  $D$  is the diagonal matrix &  $N$  is the matrix whose columns are the normalized eigen vectors of  $A$ .

#### Problems:

① Diagonalize the matrix  $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$  by

means of an orthogonal transformation?

#### Soln:

$$\text{Let } A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$



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Step 1: To find the characteristic equation:

$$\lambda^3 - c_1 \lambda^2 + c_2 \lambda - c_3 = 0 \rightarrow \textcircled{1}$$

$$c_1 = 8 + 7 + 3 = 18$$

$$c_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & -1 \end{vmatrix}$$

$$= (21 - 16) + (24 - 4) + (56 - 36) = 45$$

$$c_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & -1 & 4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14) = 0$$

subst.  $c_1, c_2, c_3$  in  $\textcircled{1}$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

Step 2: To find the eigen values:

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda = 0, \lambda^2 - 18\lambda + 45 = 0$$

$$\lambda = 3, 15$$

$$\lambda = 0, 3, 15.$$



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Step 3: To find the eigen vectors:

$$(A - \lambda I)x = 0$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \textcircled{2}$$

case (i):  $\lambda = 0$

$$\textcircled{2} \Rightarrow \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking first two rows,

$$\begin{array}{ccc} 8 & -6 & 2 \\ -6 & 7 & -4 \end{array} \begin{array}{c} x_3 \\ x_1 \\ x_2 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 8 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{10} = \frac{x_2}{26} = \frac{x_3}{-2}$$

$$x_1 = 10$$

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$



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Case (ii):  $\lambda = 3$

$$\textcircled{2} \Rightarrow \begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking first two rows,

$$\begin{array}{ccc} \begin{array}{ccc} 5 & -6 & 2 \\ -6 & 4 & -4 \end{array} & \begin{array}{ccc} 2 & -5 & 1 \\ -4 & -6 & 0 \end{array} & \begin{array}{ccc} 5 & -6 & 2 \\ -6 & 4 & -4 \end{array} \\ x_3 & x_1 & x_2 \end{array}$$

$$\begin{array}{c} x_1 \\ \left| \begin{array}{cc} -6 & 2 \\ 4 & -4 \end{array} \right| \end{array} = \begin{array}{c} -x_2 \\ \left| \begin{array}{cc} 2 & -5 \\ -4 & -6 \end{array} \right| \end{array} = \begin{array}{c} x_3 \\ \left| \begin{array}{cc} 5 & -6 \\ -6 & 4 \end{array} \right| \end{array}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16} = t$$
$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore$  The eigen vector is  $x_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Case (iii):  $\lambda = 15$

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$



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Taking first 2 rows,

$$\begin{array}{ccc} -7 & -6 & 2 \\ -6 & -8 & -4 \end{array} \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} = \begin{array}{c} -7 \\ -6 \\ -8 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & -7 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

∴ The eigen vector is  $x_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Hence the modal matrix,

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

Step 4: To find the normalised matrix N

Normalising each column vector of M,

Dividing each element of first column by 3, second column by 3 & third column by 3, we get the normalised matrix N.



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$$N = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

Step 5: calculate  $N^T A N$

$$N^T = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$N^T A N = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

$$\frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$N^T A N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} = D$$

The diagonal elements are the eigen values of A.



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② Diagonalize the matrix  $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ .

By means of an orthogonal transformation

Soln:

$$\lambda = -1, 1, 4$$

$$N = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

③  $\begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$

Soln:

$$\lambda = -4, 1, 3$$