UNIT - II

ORTHOGONAL TRANSFORMATION OF REAL SYMMETRIC MATRIX

Diagonalization of a real symmetric matrix:

Transforming a real symmetric matrix A into D by means of the transformation $\boxed{N^T A N = D}$ is known as orthogonal transformation here D is the diagonal matrix & N is the matrix whose columns are the normalized eigen vectors of A.

Problems:

① Diagonalize the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ by means of an orthogonal transformation?

Soln:

Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Step 1: To find the characteristic equation:

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0 \longrightarrow (1)$$

$$c_1 = 8 + 7 + 3 = 18$$

$$c_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= (21-16) + (24-4) + (56-36)$$

$$= 45$$

$$c_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 0$$

subs c_1, c_2, c_3 in (1)

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

Step 2: To find the eigen values:

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda = 0, \quad \lambda^2 - 18\lambda + 45 = 0$$

$$\lambda = 3, 15$$

$$\lambda = 0, 3, 15.$$

Step 3: To find the eigen vectors:

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

case (i): $\lambda = 0$

$$(2) \Rightarrow \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking first two rows,

$$\begin{matrix} 8 & & -6 & & 2 & & 8 \\ & \times & & \times & & \times & \\ -6 & & 7 & & -4 & & -6 \\ & x_3 & & x_1 & & x_2 & \end{matrix}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 8 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{\cancel{10}\ 1} = \frac{x_2}{\cancel{20}\ 2} = \frac{x_3}{\cancel{20}\ 2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case (ii): $\lambda = 3$

$$(2) \Rightarrow \begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking first two rows,

$$\begin{matrix} 5 & & -6 & & 2 & & 5 \\ & \times & & \times & & \times & \\ -6 & & 4 & & -4 & & -6 \end{matrix}$$

$$x_3 \qquad x_1 \qquad x_2$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 5 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore$ The eigen vector is $x_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

case (iii): $\lambda = 15$

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking first 2 rows,

$$\begin{matrix} -7 & & -6 & & 2 & & -7 \\ & \times & & \times & & \times & \\ -6 & & -8 & & -4 & & -6 \\ & x_3 & & x_1 & & x_2 & \end{matrix}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & -7 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$\therefore$ The eigen vector is $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Hence the model matrix,

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

Step 4: To find the normalised matrix N

Normalising each column vector of M,

Dividing each element of first column by 3, second coloumn by 3 & third coloumn by 3, we get the normalized matrix N.

$$N = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

Step 5: calculate $N^T A N$

$$N^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$N^T A N = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$N^T A N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} = D$$

The diagonal elements are the eigen values of A.

(2) Diagonalize the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by means of an orthogonal transformation.

Soln:

$\lambda = -1, 1, 4$

$$N = \begin{pmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{pmatrix}$$

(3) $\begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$

Soln:

$\lambda = -4, 1, 3$