UNIT 4 Fourier Transforms
Properties

Properties of Fourier Transform, FCT, FST:

1. Linear Property:

Fourier Transform:

$F[af(x)+bg(x)] = aF[f(x)] + bF[g(x)]$ where a and b are real numbers.

Proof:
$$F[af(x)+bg(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(x)+bg(x)]e^{isx}\,dx$$
$$= \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{isx}\,dx$$
$$= aF[f(x)] + bF[g(x)]$$

Fourier sine Transform:

$$F_s[af(x)+bg(x)] = aF_s[f(x)] + bF_s[g(x)]$$

Proof:
$$F_s[af(x)+bg(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}[af(x)+bg(x)]\sin sx\,dx$$
$$= a\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx + b\sqrt{\frac{2}{\pi}}\int_0^{\infty} g(x)\sin sx\,dx$$
$$= aF_s[f(x)] + bF_s[g(x)]$$

Similarly, Fourier cosine Transform:
$$F_c[af(x)+bg(x)] = aF_c[f(x)] + bF_c[g(x)]$$

2. Change of scale Property:

For any non zero real a, $F[f(ax)] = \frac{1}{a}F\left(\frac{s}{a}\right), a>0$

Proof:
WKT,
$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$
Now,
$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{isx}\,dx$$

Put $t = ax$

$\frac{dt}{dx} = a \quad dx = \frac{dt}{a}$ | when $x = -\infty \Rightarrow t = -\infty$, $x = \infty \Rightarrow t = \infty$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is\frac{t}{a}} \frac{dt}{a} \rightarrow i\left(\frac{s}{a}\right)t$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t/a)}\, dt$$

$$= \frac{1}{a} F\left[\frac{s}{a}\right]$$

3) Shifting Property!

i) $F[f(x-a)] = e^{ias} F(s)$

ii) $F[e^{iax} f(x)] = F(s+a)$

Proof:- $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$

i) Now, $F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$

Put $t = x-a$, $dt = dx$ | $x = -\infty \Rightarrow t = -\infty$, $x = \infty \Rightarrow t = \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist} \cdot e^{isa}\, dt$$

$$= \frac{e^{isa}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= e^{ias} F(s)$$

ii) $F[e^{iax} f(x)] = F(s+a)$

Proof! $F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx$

$$= F(s+a)$$

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4. Modulation Property:-

Fourier Transform: If $F(s)$ is the Fourier transform of $f(x)$

then $F[f(x)\cos ax] = \frac{1}{2}[F(s+a)+F(s-a)]$

Proof:- $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

Now, $F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right) e^{isx}\, dx$$

$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left[e^{i(s+a)x} + e^{i(s-a)x}\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x}\, dx\right]$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

Fourier Sine Transform:-

$F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s+a)]$

Proof:-

$F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax\, \sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\cos ax\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\frac{1}{2}\left[\sin(sx+ax) + \sin(sx-ax)\right] dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(s+a)x\, dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(s-a)x\, dx\right]$$

$$= \frac{1}{2}\left[F_s(s+a) + F(s-a)\right]$$

Fourier Cosine Transform:-

$F_c[f(x)\cos ax] = \frac{1}{2}\left[F_c(s+a) + F_c(s-a)\right]$

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5. $F\left[x^n f(x)\right] = (-i)^n \frac{d^n}{ds^n} F(s)$

Proof:- $F[s] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$

Differentiating B.s 'n' times w.r.t x.

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \frac{d^n}{ds^n} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial s^n} \left[f(x) e^{isx} \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, (ix)^n e^{isx} dx$$

$$= i^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, x^n e^{isx} dx$$

$$= i^n F\left[x^n f(x)\right]$$

$$\Rightarrow F\left[x^n f(x)\right] = \frac{1}{(i)^n} \frac{d^n}{ds^n} F(s)$$

$$= (-i)^n \frac{d^n F(s)}{ds^n}$$

6) i) $F\left[f'(x)\right] = -isF(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

ii) $F\left[f^n(x)\right] = (-i)^n s^n F(s)$ if $f(x), f'(x), \ldots f^{(n-1)}(x) \to 0$ as $x \to \pm\infty$

7) $F\left[\overline{f(x)}\right] = \overline{F(-s)}$

8) i) $F_s\left[x f(x)\right] = -\frac{d}{ds} F_c\left[f(x)\right]$

ii) $F_c\left[x f(x)\right] = \frac{d}{ds} F_s\left[f(x)\right]$