State and prove distributive inequality of Lattice:

Distributive Inequality Lattice:

Let $(L, \wedge, \vee)$ be a gn lattice

For any $a, b, c \in L$, the following holds

1) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

2) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$.

Proof:

Claim ①

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

From the defn of LUB, it is obvious that $a \leq a \vee b$.

$a \vee b$ is an upper bound of $\{a, b \wedge c\}$

Hence $a \vee b \geq a \vee (b \wedge c)$ ——— (1)

From the defn of LUB, it is obvious

that $a \leq a \vee c$

$b \wedge c \leq c \leq a \vee c$.

$\Rightarrow b \wedge c \leq a \vee c$

$\Rightarrow a \vee c$ is an upper bound of $\{a, b \wedge c\}$

$a \vee c \geq a \vee (b \wedge c)$ ——— (2)

From (1) & (2) $a \vee (b \wedge c)$ is a LB of

$\{a \vee b, a \vee c\}$

$\therefore a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Claim: 2

$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

WKT, $a \geq a \wedge b$. ——— (1)

and $b \vee c \geq b \geq a \wedge b$

From (1) & (2),

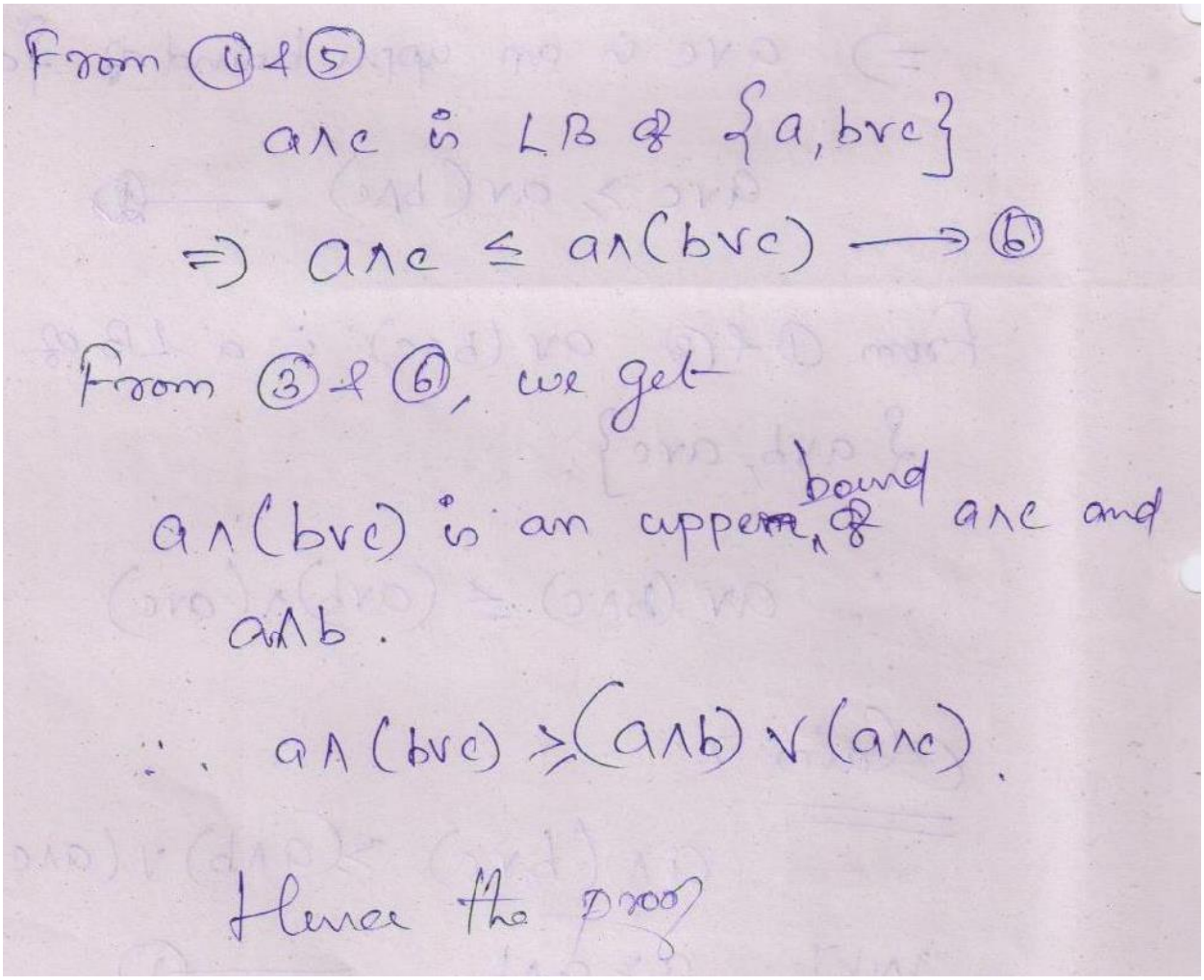
$a \wedge b$ is an LB of $\{a, b \vee c\}$

$a \wedge b \leq a \wedge (b \vee c)$ ——— (3)

WKT, $a \geq a \wedge c$ ——— (4)

and $b \vee c \geq c \geq a \wedge c$

$\Rightarrow b \vee c \geq a \wedge c$ ——— (5)


From (4) & (5)
a∧c is LB of {a, b∨c}
⇒ a∧c ≤ a∧(b∨c) ⟶ (6)
From (3) & (6), we get
a∧(b∨c) is an upper bound of a∧c and a∧b.
∴ a∧(b∨c) ≥ (a∧b)∨(a∧c).
Hence the proof