Problems:

1. Show that the set $G = \{1, -1, i, -i\}$ consisting of the 4^{th} roots of unity is a commutative group under multiplication.

Soln.:

Multiplication (Cayley) Table

$*$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

i). Closure: Now $1, -1 \in G$, $+1 * -1 = -1 \in G$

$\therefore$ G is closed.

ii). Associative: $1, -1, i \in G$ $\quad (1 * -1) * i = -i \in G$

$1 * (-1 * i) = -i \in G$

$\therefore (1 * -1) * i = 1 * (-1 * i)$

It satisfies the associativity.

iii). Identity Elt.: For $1, -1, i, -i \in G$

$1 * 1 = 1$, $-1 * +1 = -1$, $i * +1 = i$, $-i * 1 = -i$

$\therefore$ 1 is the identity elt.

iv). Inverse elt.:

Inverse of -1 is -1 ie., $-1 * -1 = 1 \in G$

Inverse of 1 is 1 ie., $1 * 1 = 1 \in G$

Inverse of i is $-i$ ie., $i * -i = 1 \in G$
Inverse of $-i$ is i ie., $-i * i = 1 \in G$

v). commutative: $i, -i \in G$ $\quad i * -i = 1 \in G$
$-i * i = 1 \in G$

$\Rightarrow i * -i = -i * i$

$\therefore G$ is commutative group under multiplication.

2) Prove that the set $A = \{1, \omega, \omega^2\}$ is an abelian group of order 3 under usual multiplication where $1, \omega, \omega^2$ are cube roots of unity and $\omega^3 = 1$

Soln.
composition table

$*$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

i). closure:
All the elts in the above table are the elts. of A. Hence A is closed under multiplication.

ii). Associative:
$(1 * \omega) * \omega^2 = \omega^3 = 1 \in A$
$1 * (\omega * \omega^2) = \omega^3 = 1 \in A$
It satisfies the associative property.
$(1 * \omega) * \omega^2 = 1 * (\omega * \omega^2)$

iii). Identity elt.: $1, \omega, \omega^2 \in A$
$1 * 1 = 1, \quad 1 * \omega = \omega, \quad \omega^2 * 1 = \omega^2$
1 is the identity elt. of A

iv). Inverse elt.
Inverse of 1 is 1 ie., $1 * 1 = 1 \in A$
ω is ω^2 ie., $\omega * \omega^2 = \omega^3 = 1 \in A$
ω^2 is ω ie., $\omega^2 * \omega = \omega^3 = 1 \in A$

v). commutative :

$1 * \omega = \omega \in A$

$\omega * 1 = \omega \in A$

Hence $(A, *)$ is an abelian group.

3). Let I be the set of integers. Let Z_m be the set of equivalence classes generated by the equivalence relation " congruence modulo m" for any +ve integer m. Then $(Z_m, +_m)$ and $(Z_m, \times_m)$ are monoids.

Soln.

For $[i], [j] \in Z_m$

a). $+_m$ is defined as $[i] +_m [j] = [(i+j)(\text{mod } m)]$

b). $\times_m$ is defined as $[i] \times_m [j] = [(i \times j)(\text{mod } m)]$

The composition table for $m = 5$ is given as

$(Z_5, +_5)$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$(Z_5, \times_5)$

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

i). closure property :

In the above table $(Z_5, +_5)$ and $(Z_5, \times_5$ satisfies closure property.

ii). Associative :

clearly, $(Z_5, +_5)$ and $(Z_5, \times_5)$ satisfies associative property.

iii). Identity elt. :

$[0]$ is the identity elt. w.r. to $+_m$

$[1]$ is the " " " " $\times_m$

$\therefore$ $(Z_m, +_m)$ and $(Z_m, \times_m)$ are monoids.

4] Show that $(Q^+, *)$ is an abelian group where $*$ is defined by $a*b = \frac{ab}{2}, \forall a,b \in Q^+$

Soln.

i). For $a, b \in Q^+ \Rightarrow a*b = \frac{ab}{2} \in Q^+$

$\therefore$ Q^+ is closed

ii). For $a, b, c \in Q^+$. Then $a*(b*c) = a*\frac{bc}{2}$

$$= \frac{abc}{4} \rightarrow (1)$$

$$(a*b)*c = \frac{ab}{2}*c = \frac{abc}{4} \rightarrow (2)$$

From (1) and (2),

$$a*(b*c) = (a*b)*c$$

iii). Identity:

Let $a \in Q^+$. Then $\exists\, e \in Q^+$ such that

Now $a*e = a$

$$\frac{ae}{2} = a \Rightarrow e = 2$$

iv). Inverse elt:

Let $a \in Q^+$. Then $\exists\, a^{-1} \in Q^+$ such that

$a * a^{-1} = e$

$$\frac{a\,a^{-1}}{2} = 2 \Rightarrow a^{-1} = \frac{4}{a}$$

v). Commutative:

Let $a, b \in Q^+$. Then $a*b = \frac{ab}{2}$

and $b*a = \frac{ba}{2}$

$\therefore$ $a*b = b*a$

Hence $(Q^+, *)$ is an abelian group.

Q]. Let G denote the set of all matrices of the form $\begin{bmatrix} x & x \\ x & x \end{bmatrix}$ where $x \in R$. Prove that G is a group under matrix multiplication.

Soln.

i). Closure:

Let $A, B \in G$

Let $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$; $B = \begin{bmatrix} y & y \\ y & y \end{bmatrix}$

Then $AB = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} y & y \\ y & y \end{bmatrix} = \begin{bmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{bmatrix}$

$= \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} \in G$

ii). Associative:

Matrix multiplication is Associative.

iii). Identity elt.:

Let $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$. Then $\exists\ E = \begin{bmatrix} e & e \\ e & e \end{bmatrix} \ni AE = A$

Now, $\begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} e & e \\ e & e \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$

$\begin{bmatrix} 2xe & 2xe \\ 2xe & 2xe \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$

$2xe = x \Rightarrow e = 1/2$

Hence $E = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is the identity elt. of G.

iv). Inverse elt.:

Let $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$. Then $\exists\ A^{-1} = \begin{bmatrix} x_0 & x_0 \\ x_0 & x_0 \end{bmatrix} \ni$

$AA^{-1} = E \Rightarrow \begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} x_0 & x_0 \\ x_0 & x_0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$\begin{bmatrix} 2xx_0 & 2xx_0 \\ 2xx_0 & 2xx_0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$2xx_0 = \frac{1}{2} \Rightarrow x_0 = 1/4x$

Hence $A^{-1} = \begin{bmatrix} 1/4x & 1/4x \\ \frac{1}{4x} & \frac{1}{4x} \end{bmatrix}$ is the inverse of A

Hence G is a group under matrix multiplication.

HW. S.T. $(R-\{1\}, *)$ is an abelian group, where $*$ is defined by $a*b = a+b+ab$, $\forall a, b \in R$