Normal subgroup:

Let H be a subgroup of G under $*$. Then H is said to be a normal subgroup of G, for evy. $x \in G$ and for $h \in H$ if

$$x * h * x^{-1} \in H$$

and $x * H * x^{-1} \subseteq H$

Alternatively, a subgroup H of G is called a normal subgroup of G if $x * h = h * x$, $\forall x \in G$.

Theorem: 1

The intersection of any 2 normal subgroups is a normal subgroup.

Proof:

Let H and K be the 2 normal subgroups.

$\Rightarrow H$ and K are subgroups of G.

$\Rightarrow H \cap K$ is a subgroup of G (Already proved)

Now we've to prove that $H \cap K$ is normal.

Let $x \in G$ and $h \in H \cap K$

$x \in G$ and $h \in H$ and $h \in K$

$\Rightarrow x \in G$, ~~and~~ $h \in H$ and $x \in G$, $h \in K$

$\Rightarrow \underbrace{x * h * x^{-1} \in H}_{(1)}$ and $\underbrace{x * h * x^{-1} \in K}_{(2)}$ ($\because H$ and K are normal subgroups)

From (1) and (2), we get

$$x * h * x^{-1} \in H \cap K$$

$\Rightarrow H \cap K$ is a normal subgroup of G.

Theorem 2:

Let G and G' be any two groups with identity elt. e and e' resply. If $f: G \to G'$ be a homomorphism, then $Ker(f)$ is a normal subgroup.

Proof:

Let $K = Ker\ f = \{x \in G / f(x) = e'\}$

We know that $Ker(f)$ is a subgroup of G

Now we've to prove that $Ker(f)$ is normal.

To prove $x * h * x^{-1} \in K$

Let $x \in G$ and $h \in K$

$\therefore f(x * h * x^{-1}) = f(x) * f(h) * f(x^{-1})$ $\quad \because f$ is a homo.

$= f(x) * e' * f(x^{-1})$ $\quad [\because h \in K = Ker f]$

$= f(x) * f(x^{-1})$

$= f(x * x^{-1})$

$= f(e)$

$= e'$

$f(x * h * x^{-1}) = e'$

$\Rightarrow x * h * x^{-1} \in K$

$\therefore K = Ker\ f$ is a normal subgroup of G.

Theorem: 3 Fundamental Theorem of Homomorphism

Every homomorphic image of a group G is isomorphic to some quotient group of G.

(or)

Let $f: G \to G'$ be a onto homomorphism of groups with kernel K. Then $G/K \simeq G'$.

Proof:

Let $f: G \to G'$ be a homomorphism

Let K be the kernel of this homo.

Clearly K is a normal subgroup of G.

To prove G/K is isomorphic $G/K \simeq G'$.

i). To prove ϕ is well defined.

Let $\phi: G/K \to G'$ by $\phi(K*a) = f(a)$.

Consider, $K*a = K*b$

$\Rightarrow a*b^{-1} \in K$

$\Rightarrow f(a*b^{-1}) = e'$

$f(a) * f(b^{-1}) = e'$

$f(a) * [f(b)]^{-1} = e'$

$f(a) * [f(b)]^{-1} * f(b) = e' * f(b)$

$f(a) * e = e' * f(b)$

$f(a) = f(b)$

$\phi(K*a) = \phi(K*b)$

$\therefore \phi$ is well defined.

ii). To prove ϕ is 1-1.

ie., $\phi(K*a) = \phi(K*b) \Rightarrow K*a = K*b$

consider $\phi(K*a) = \phi(K*b)$

$$f(a) = f(b)$$

$$f(a) * f(b^{-1}) = f(b) * f(b^{-1})$$

$$f(a * b^{-1}) = f(b * b^{-1})$$

$$= f(e)$$

$$= e'$$

$$\Rightarrow a * b^{-1} \in K$$

$$K * a = K * b$$

$\therefore$ ϕ is 1–1.

iii). To prove ϕ is onto,

Let $b \in G'$

Since f is onto $\exists$ an elt. $a \in G$ such that

$f(a) = b$.

$$\Rightarrow f(a) = \phi(K * a) = b$$

$\therefore$ ϕ is onto.

iv). To prove ϕ is a homo.

Now,

$$\phi(K * a * K * b) = \phi(K * a * b)$$

$$= f(a * b)$$

$$= f(a) * f(b)$$

$$= \phi(K * a) * \phi(K * b)$$

$\therefore$ ϕ is a homo.

Since ϕ is 1–1 & onto, homo.

$\therefore$ ϕ is an isomorphism b/w G/K and G'

$$\Rightarrow G/K \simeq G'$$