UNIT 4 Fourier Transforms
Sine and Cosine Transform

(4) Find the fourier sine & cosine transform of e^{-ax} and deduce that inverse fourier transform & , parsevals identity

Sine transform:

$$F_S(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$$

Inverse Fourier sine Transform:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right) \sin sx \, ds$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds$$

$$\int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds = \frac{\pi}{2} e^{-ax}$$

Parseval's Identity :-

$$\int_0^\infty (f(x))^2 dx = \int_0^\infty (F_S(s))^2 ds$$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right)\right)^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi} \int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds$$

$$\left[\frac{e^{-\infty}}{-2a} - \frac{e^0}{-2a}\right] = \frac{2}{\pi} \int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds$$

$$\frac{2}{\pi} \int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds = \frac{1}{2a}$$

$$\int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds = \frac{\pi}{4a}$$

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Cosine Transform:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx = \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

Inversion:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right] \cos sx \, ds$$

$$e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{a}{s^2+a^2} \cos sx \, ds$$

$$\Rightarrow \int_0^\infty \frac{a}{s^2+a^2} \cos sx \, dx = \frac{\pi}{2} e^{-ax}$$

Parseval's Identity:-

$$\int_0^\infty (f(x))^2 dx = \int_0^\infty (F(s))^2 ds$$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right)\right)^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi} \int_0^\infty \left(\frac{a}{a^2+s^2}\right)^2 ds$$

$$\int_0^\infty \left(\frac{a}{s^2+a^2}\right)^2 ds = \frac{\pi}{2}\left(\frac{1}{2a}(0-1)\right)$$

$$= \frac{\pi}{4a}.$$

5. Find the Fourier Cosine Transform of $\frac{e^{-ax}}{x}$ and hence find $F_c\left[\frac{e^{-ax} - e^{-bx}}{x}\right]$

$$F_c[s] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx$$

$$\frac{d}{ds} F_c[s] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx\right]$$

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$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\cos sx\right) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x}(-x\sin sx)\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\sin sx\, dx$$

$$\frac{d}{ds} F_c[s] = -\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2} \qquad \because \int_0^\infty e^{-ax}\sin bx\, dx = \frac{b}{a^2+b^2}$$

Integrating, we get

$$F_c[s] = -\sqrt{\frac{2}{\pi}} \int \frac{s}{s^2+a^2}\, ds$$

$$= -\sqrt{\frac{2}{\pi}}\frac{1}{2} \int \frac{2s}{s^2+a^2}\, ds$$

$$F_c[s] = -\frac{1}{\sqrt{2\pi}} \log(s^2+a^2)$$

Similarly, $F_c\left[\frac{e^{-bx}}{x}\right] = \frac{-1}{\sqrt{2\pi}} \log(s^2+b^2)$

Now,

$$F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$$

$$= \frac{-1}{\sqrt{2\pi}} \log(s^2+a^2) + \frac{1}{\sqrt{2\pi}} \log(s^2+b^2)$$

$$= \frac{1}{\sqrt{2\pi}}\left[\log(s^2+b^2) - \log(s^2+a^2)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

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1) Find the FST and FCT of $x e^{-ax}$

By property,

$$F_s[x f(x)] = -\frac{d}{ds} F_c[f(x)]$$

$$F_s[x e^{-ax}] = \frac{-d}{ds} F_c[e^{-ax}]$$

$$= \frac{-d}{ds} \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

$$F_s[x e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{2as}{(a^2+s^2)^2}$$

and $F_c[x f(x)] = \frac{d}{ds} F_s(f(x))$

$$F_c[x e^{-ax}] = \frac{d}{ds}[F_s(e^{-ax})]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

Identity Property:-

i) $\int_0^\infty f(x)g(x)\,dx = \int_0^\infty F_c(s)\,G_c(s)\,ds$

ii) $\int_0^\infty f(x)g(x)\,dx = \int_0^\infty F_s(s)\,G_s(s)\,ds$

(iii) $\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s(s)|^2 ds$

iv) $\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(s)|^2 ds$

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Find the Fourier Sine Transform of $\frac{e^{-ax}}{x}$ & hence find $F_S\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$

$$F_S\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx.$$

Diff w.r.t s,

$$\frac{d}{ds}F_S\left[\frac{e^{-ax}}{x}\right] = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial s}(\sin sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{\cancel{x}}\cdot\cos sx\cdot\cancel{x}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

Integrating both sides by 's'

$$\int\frac{d}{ds}F_S\left(\frac{e^{-ax}}{x}\right) = \sqrt{\frac{2}{\pi}}\,a\int\frac{1}{s^2+a^2}\,ds$$

$$\int\frac{1}{s^2+a^2}\,ds = \frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)$$

$$= \sqrt{\frac{2}{\pi}}\cdot a\left(\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right)$$

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$$

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Similarly, $F_s\left(\frac{e^{-bx}}{x}\right) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{b}\right)$

$$F_s\left[\frac{e^{-ax} - e^{-bx}}{x}\right] = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)\right]$$

Find Fourier sine Transform & Fourier cosine Transform of $e^{-a|x|}$. Hence show that

i) $\displaystyle\int_0^\infty \frac{\cos sx}{s^2+a^2}\,dx = \frac{\pi}{2a} e^{-as}$

ii) $\displaystyle\int_0^\infty \frac{x \sin sx}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-as}.$

By Fourier sine Transform,

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right)$$

Inverse fourier sine transform of $f(x)$ is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[f(x)] \sin sx \, ds$$

$(x \leftrightarrow s)$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right) \sin sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s \sin sx}{s^2+a^2} \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{s^2+a^2} \, ds$$

Put $s = x$ & $x = s$

$\displaystyle\int_0^\infty \frac{x \sin xs}{x^2+a^2}\,dx.$

$$\frac{\pi}{2} e^{-ax} = \int_0^\infty \frac{s \sin sx}{s^2+a^2} \, ds$$

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Put $s = x$

$$\int_0^\infty \frac{x \sin sx}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-as}$$

Fourier cosine transform,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\,dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right)$$

Inverse fourier transform of

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx\,ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right) \cos sx\,ds$$

$$e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{s^2+a^2}\,ds$$

$$\frac{\pi}{2a} e^{-as} = \int_0^\infty \frac{\cos sx}{s^2+a^2}\,dx.$$