



UNIT 4 Fourier Transforms
Fourier transform pair

Fourier Transform Pair:

The Fourier transform of $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \rightarrow \textcircled{1} \quad \begin{array}{l} \text{Complex Fourier transform} \\ \text{of } f(x) \end{array}$$

Then the function $f(x)$ is the Inverse Fourier Transform of $F(s)$ is given by,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds \rightarrow \textcircled{2}$$

The above eqns $\textcircled{1}$ and $\textcircled{2}$ are jointly called Fourier

Transform pair,

$$f(x) = F^{-1}[F(s)] = F^{-1}[F[f(x)]]$$

1. Find the Fourier transform of the function

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

and hence deduce

$$\text{that } \int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$$

Fourier Transform is

$$\begin{aligned} F(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a 1 (\cos sx + i \sin sx) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a \cos sx dx = \frac{2}{\sqrt{2\pi}} \int_0^a \cos sx dx \end{aligned}$$



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$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - \frac{\sin 0}{s} \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin sa}{s} \right)$$

Inverse fourier transform is $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$

$$1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s} (\cos sx - i \sin sx) ds$$

$$1 = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} \left[\int_{-\infty}^{\infty} \frac{\sin sa}{s} \cos sx ds \right. \\ \left. - i \int_{-\infty}^{\infty} \frac{\sin sa}{s} \sin sx ds \right]$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa}{s} \cos sx ds$$

$$1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin sa}{s} \cos sx ds$$

$$\frac{\pi}{2} = \int_0^{\infty} \frac{\sin sa}{s} \cos sx ds$$

Put $x=0$, $a=1$

$$\frac{\pi}{2} = \int_0^{\infty} \frac{\sin s}{s} ds \quad \text{put } s=t, ds=dt$$

$$\frac{\pi}{2} = \int_0^{\infty} \frac{\sin t}{t} dt$$



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① Find the FT of $f(x) = \begin{cases} x & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a \end{cases}$

$$f(x) = \begin{cases} x & \text{if } -a \leq x \leq a \\ 0 & \text{if } -\infty < x < -a \quad \& \quad a < x < \infty \end{cases}$$

$$\begin{aligned} F(s) = F[f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a x [\cos sx + i \sin sx] dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^a x \cos sx dx + i \int_{-a}^a x \sin sx dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[2i \int_0^a x \sin sx dx \right] \\ &= \frac{2i}{\sqrt{2\pi}} \left[x \left(\frac{\cos sx}{s} \right) - 1 \left(\frac{-\sin sx}{s^2} \right) + 0 \right]_0^a \\ &= \frac{2i}{\sqrt{2\pi}} \left[\left(-\frac{a \cos sa}{s} + \frac{\sin sa}{s^2} \right) - 0 \right] \\ &= \frac{2i}{\sqrt{2\pi}} \left[\frac{\sin sa - sa \cos sa}{s^2} \right] \end{aligned}$$

③ Find the Fourier transform of the function

$$f(x) = \begin{cases} 1-x^2 & , |x| < 1 \\ 0 & , |x| > 1 \end{cases} \quad \& \quad \text{Hence deduce that}$$

$$\begin{aligned} \int_0^{\infty} \frac{\sin s}{s^3} - \frac{s \cos s}{s^3} \cdot \cos \frac{s}{2} ds &= \frac{3\pi}{16} \\ F(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (1-x^2) e^{isx} dx. \end{aligned}$$



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By using Bernoulli's formula,

$$u = 1 - x^2$$

$$u' = -2x$$

$$u'' = -2$$

$$v = e^{isx}$$

$$v_1 = \frac{e^{isx}}{is}$$

$$v_2 = \frac{e^{isx}}{(is)^2} = -\frac{e^{isx}}{s^2}$$

$$v_3 = \frac{e^{isx}}{(is)^3} = -\frac{e^{isx}}{is^3}$$

$$= \frac{1}{\sqrt{2\pi}} \left[(1-x^2) \frac{e^{isx}}{is} + 2x \left(-\frac{e^{isx}}{s^2} \right) - 2 \left(\frac{e^{isx}}{is^3} \right) \right]_-$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[0 + 2 \left(-\frac{e^{isx}}{s^2} \right) + 2 \frac{e^{isx}}{is^3} \right] - \left[0 + 2 \frac{e^{-is}}{s^2} + 2 \frac{e^{-is}}{is^3} \right] \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[-\frac{e^{is}}{s^2} + \frac{e^{is}}{is^3} - \frac{e^{-is}}{s^2} - \frac{e^{-is}}{is^3} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[- \left(\frac{e^{is} + e^{-is}}{s^2} \right) + \frac{e^{is}}{is^3} - \frac{e^{-is}}{is^3} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[- \left[\frac{\cos s + i \sin s + \cos s - i \sin s}{s^2} \right] \right]$$

$$+ \left[\frac{\cos s + i \sin s - \cos s + i \sin s}{is^3} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-2 \cos s}{s^2} + \frac{2i \sin s}{is^3} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{2 \sin s - 2 \cos s}{s^3} \right]$$



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Inverse Fourier transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$1-x^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2\sqrt{2}}{\sqrt{\pi}} \left[\frac{\sin s - s \cos s}{s^3} \right] [\cos sx - i \sin sx] ds$$

$$1-x^2 = \frac{2}{\pi} \left[\int_{-\infty}^{\infty} \left(\frac{\sin s - s \cos s}{s^3} \right) \cos sx ds - i \int_{-\infty}^{\infty} \left(\frac{\sin s - s \cos s}{s^3} \right) \sin sx ds \right]$$

\downarrow even \downarrow odd

$$1-x^2 = \frac{4}{\pi} \int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos sx ds$$

Put $x = \frac{1}{2}$.

$$1 - \frac{1}{4} = \frac{4}{\pi} \int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos \frac{s}{2} ds$$

$$\frac{3}{4} = \frac{4}{\pi} \int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos \frac{s}{2} ds$$

$$\int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos \frac{s}{2} ds = \frac{3\pi}{16}$$