UNIT 5 Z - Transforms and Difference equations
Elementary Properties

If $Z[f(n)] = F(z)$ then $Z^{-1}[F(z)] = f(n)$

1. $Z[a^n] = \frac{z}{z-a} \Rightarrow Z^{-1}\left[\frac{z}{z-a}\right] = a^n$

2. $Z[(-a)^n] = \frac{z}{z+a} \Rightarrow (-a)^n = Z^{-1}\left[\frac{z}{z+a}\right]$

3. $Z(n) = \frac{z}{(z-1)^2}$

4. $Z[n a^n] = \frac{az}{(z-a)^2}$

5. $Z[n a^{n-1}] = \frac{z}{(z-a)^2}$

6. $Z[a^{n-1}] = \frac{1}{z-a}$

7. $Z[(-a)^{n-1}] = \frac{1}{z+a}$

8. $Z[n(-a)^{n-1}] = \frac{z}{(z+a)^2}$

9. $Z[(n-1)a^{n-2}] = \frac{1}{(z-a)^2}$

10. $Z[(n-1)(-a)^{n-2}] = \frac{1}{(z+a)^2}$

11. $Z\left[\frac{1}{n+1}\right] = -z\log(1-1/z)$

12. $Z\left[\cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+1}$ 13. $Z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$

14. $Z\left[\frac{1}{n!}\right] = e^{1/z}$

15. $Z[n^2] = \frac{z(z+1)}{(z-1)^3}$

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1. Find the z-transform of 1 (or) $Z(1)$

or Prove that $Z(1) = \frac{z}{z-1}$, $|z| > 1$

We know that, $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z(1) = \sum_{n=0}^{\infty} (1) z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1} = \frac{z}{z-1}$$

2. Find $Z((-1)^n)$

We know that, $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z[(-1)^n] = \sum_{n=0}^{\infty} (-1)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{z}\right)^n$$

$$= 1 + \left(\frac{-1}{z}\right)^1 + \left(\frac{-1}{z}\right)^2 + \left(\frac{-1}{z}\right)^3 + \cdots$$

$$= 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots$$

$$= \left(1 + \frac{1}{z}\right)^{-1} = \left(\frac{z+1}{z}\right)^{-1}$$

$$Z[(-1)^n] = \frac{z}{z+1}$$

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Properties :-

1. Linear Property:

$$Z[af(n) + bg(n)] = aF(z) + bG(z), \quad Z[f(n)] = F(z)$$
$$Z[g(n)] = G(z)$$

'a' and 'b' are constants

Proof:-

$$Z[af(n) + bg(n)] = \sum_{n=0}^{\infty} [af(n) + bg(n)] z^{-n}$$
$$= \sum_{n=0}^{\infty} af(n) z^{-n} + \sum_{n=0}^{\infty} bg(n) z^{-n}$$
$$= aZ[f(n)] + bZ[g(n)]$$
$$= aF(z) + bG(z)$$

2. First Shifting Property:-

If $Z[f(t)] = F(z)$, then $Z[e^{-at} f(t)] = F[ze^{aT}]$

(or)

$$Z[e^{-at} f(t)] = \{F(z)\}_{z \to ze^{aT}}, \quad Z\{e^{at} f(t)\} = \{F(z)\}_{z \to ze^{-aT}}$$

Proof:-

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$
$$Z[e^{-at} f(t)] = \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}$$
$$= \sum_{n=0}^{\infty} f(nT) (ze^{aT})^{-n}$$
$$= F[ze^{aT}]$$

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3. Change of scale:

If $z[f(n)] = F(z)$ then $z[a^n f(n)] = F\left(\frac{z}{a}\right)$

Proof:- $z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$\Rightarrow z[a^n f(n)] = \sum_{n=0}^{\infty} a^n f(n) z^{-n} = \sum_{n=0}^{\infty} f(n) \left(\frac{z}{a}\right)^{-n}$

$= F\left(\frac{z}{a}\right) = \{F(z)\}_{z \to \frac{z}{a}}$

4. Second Shifting property:

If $z[f(n)] = F(z)$ then $z[f(n+1)] = zF(z) - zf(0)$

5). Differentiation in z-domain:

i) $z[nf(n)] = -z \frac{d}{dz}\{F(z)\}$, where $F(z) = z\{f(n)\}$

ii) $z[nf(t)] = -z \frac{d}{dz}\{F(z)\}$, where $F(z) = z[f(t)]$

Proof:- $F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$

$\frac{d}{dz} F(z) = \frac{d}{dz} \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} f(n)(-n) z^{-n-1}$

$= \frac{-1}{z} \sum_{n=0}^{\infty} n f(n) z^{-n} = \frac{-1}{z} z[nf(n)]$

$z[nf(n)] = -z \frac{d}{dz} F(z)$

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1. Find $Z[e^{-iat}]$

$$Z[e^{-iat}] = Z[e^{-iat}(1)] = [Z(1)]_{z \to ze^{iaT}}$$

$$= \left[\frac{z}{z-1}\right]_{z \to ze^{iaT}}$$

$$= \frac{ze^{iaT}}{ze^{iaT}-1}$$

2. Find $Z[\cos at]$ and $Z[\sin at]$

$$Z[e^{iaT}] = Z[e^{iat}(1)] = [Z(1)]_{z \to ze^{-iaT}}$$

$$= \left[\frac{z}{z-1}\right]_{z \to ze^{-iaT}}$$

$$= \frac{ze^{-iaT}}{ze^{-iaT}-1}$$

$\div$ by e^{-iaT}

$$= \frac{ze^{-iaT}/e^{-iaT}}{\frac{ze^{-iaT}-1}{e^{-iaT}}} = \frac{z}{z-e^{+iaT}}$$

$$Z[\cos at + i\sin at] = \frac{z}{z-(\cos aT + i\sin aT)}$$

$$= \frac{z}{(z-\cos aT)-i\sin aT} \times \frac{(z-\cos aT)+i\sin aT}{(z-\cos aT)+i\sin aT}$$

$$= \frac{z(z-\cos aT)+iz\sin aT}{(z-\cos aT)^2+(\sin aT)^2}$$

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$$z\cos aT + iz\sin aT = \frac{z(z-\cos aT) + iz\sin aT}{z^2 - 2z\cos aT + 1}$$

Equating real & imaginary parts we get,

$$Z[\cos aT] = \frac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1},$$

$$Z[\sin aT] = \frac{z\sin at}{z^2 - 2z\cos aT + 1}$$