SINE TRANSFORM :

The Fourier sine transform of a function $f(x), 0<x<\infty$ is defined as $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

The Inverse Fourier sine transform of $F_s(s)$ is defined as $f(x) = F^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$.

Parseval's Identity : If $F_s(s)$ is the Fourier transform of $f(x)$ then $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_s(s)]^2 ds$.

COSINE TRANSFORM :

The Fourier cosine transform of a function $f(x), 0<x<\infty$ is defined as $F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$.

The Inverse Fourier cosine transform of $F_c(s)$ is defined as $f(x) = F^{-1}[F_c(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$.

If $F_c(s)$ is the Fourier transform of $f(x)$ then parseval's Identity is $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_c(s)]^2 ds$

1) Find the Fourier sine transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$$

soln: WKT $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{s}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s}\right]$$

2) Find the Fourier sine transform of $1/x$.

soln: WKT $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x} \, dx$$

putting $\theta = sx \Rightarrow d\theta = s\,dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta/s} \, d\theta/s$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta} \, d\theta \quad \left[\because \int_0^\infty \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}$$

$$= \sqrt{\pi/2}$$

3) Find the Fourier cosine transform of $2e^{-3x}+3e^{-2x}$.

Soln:

WKT $F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \left(2e^{-3x}+3e^{-2x}\right)\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[2\left[\frac{3}{s^2+9}\right]+3\left[\frac{2}{s^2+4}\right]\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{6}{s^2+9}+\frac{6}{s^2+4}\right]$$

4) Find the Fourier cosine transform of $f(x)=\begin{cases}\cos x, & \text{if } 0<x<a\\ 0, & \text{if } x\geq a\end{cases}$

Soln:

WKT $F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$=\sqrt{\frac{2}{\pi}}\int_0^a \cos x\,.\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^a \frac{1}{2}\left[\cos(s+1)x+\cos(s-1)x\right]dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^a\left[\cos(s+1)x+\cos(s-1)x\right]dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)x}{s+1}+\frac{\sin(s-1)x}{s-1}\right]$$

5) Find the Fourier sine & cosine transform of e^{-ax} and deduce that inverse Fourier transform & parseval's identity

soln:

Sine transform :

WKT $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$

$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2} \right]$

Inverse Transform:

WKT $f(x) = F^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sin sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds$$

$$\frac{\pi}{2} e^{-ax} = \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds$$

Parseval's Identity:

WKT $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_s(s)]^2 ds$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]\right]^2 ds$$

$$\int_0^\infty e^{-2ax} dx = \frac{2}{\pi} \int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{1}{2a} = \frac{2}{\pi} \int_0^\infty \left[\frac{s}{s^2+a^2}\right]^2 ds$$

$$\frac{\pi}{4a} = \int_0^\infty \left[\frac{s}{s^2+a^2}\right]^2 ds$$

Cosine Transform:

WKT $$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

Inverse Transform:

WKT $$f(x) = F^{-1}[F_c(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right] \cos sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{a}{s^2+a^2} \cdot \cos sx \, ds$$

put $x = 0$

$$1 = \frac{2}{\pi} \int_0^\infty \frac{a}{s^2+a^2} \cos s(0) \, ds$$

$$\frac{\pi}{2} = \int_0^\infty \frac{a}{s^2+a^2} \, ds.$$

Parseval's Identity:

WKT $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_c(s)]^2 ds$

$$\int_0^\infty [e^{-ax}]^2 dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}\right]^2 ds$$

$$\int_0^\infty e^{-2ax} dx = \frac{2}{\pi} \int_0^\infty \left[\frac{a}{s^2+a^2}\right]^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty ; \frac{1}{2a} = \frac{2}{\pi} \int_0^\infty \left[\frac{a}{s^2+a^2}\right]^2 ds$$

$$\frac{\pi}{4a} = \int_0^\infty \left[\frac{a}{s^2+a^2}\right]^2 ds .$$