UNIT 2

SHORT ANSWER

Problem 1: Write the equation of the tangent plane at (1, 5, 7) to the sphere $(x-2)^2+(y-3)^2+(z-4)^2=14$.

Solution:
The equation of the tangent plane to the sphere
$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ at (x_1, y_1, z_1) is

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0 \quad (1)$$

Given : $(x-2)^2+(y-3)^2+(z-4)^2=14$

$(x^2-4x+4)+(y^2-6y+9)+(z^2-8z+16)=14$

$x^2+y^2+z^2-4x-6y-8z+29-14=0$

Here 2u = -4, 2v = -6, 2w = -8, d = 15
$x_1=1$, $y_1=5$, $z_1=7$

$(1)\Rightarrow x(1)+y(5)+z(7)+(-2)(x+1)+(-3)(y+5)+(-4)(z+7)+15=0$

$x+5y+7z-2x-2-3y-15-4z-28+15=0$

$-x+2y+3z-30=0$

i.e., $x-2y-3z+30=0$

Problem 2: Test whether the plane x = 3 touches the sphere $x^2+y^2+z^2=9$.

Solution: The condition that the plane $lx+my+nz=p$ to touch the sphere
$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is

$$\frac{l(-u)+m(-v)+n(-w)-p}{\sqrt{l^2+m^2+n^2}}=\sqrt{u^2+v^2+w^2-d}$$

i.e., $(lu+mv+nw+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d) \quad (1)$

$u=0, v=0, w=0, l=1, m=0, n=0, p=3, d=-4$

Hence $(1)\Rightarrow(0+0+3)^2=(1+0+0)(0+0+0+9)$

i.e., $3^2=9$

The plane x=3 touches the sphere $x^2+y^2+z^2=9$.

Problem 3: Find the equation of the sphere which has its centre at (-1, 2, 3) and touches the plane 2x-y+2z = 6

Solution: Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad (1)$$

Given: $-u=-1, \quad -v=2, \quad -w=3$

$u = 1, \quad v = -2, \quad w = -3$

$$\therefore (1) \Rightarrow x^2 + y^2 + z^2 + 2x - 4y - 6z + d = 0 \qquad (2)$$

To find d:

Since the plane 2x-y+2z = 6 touches the sphere whose centre is (-1, 2, 3).

The radius of the sphere is equal to the length of the perpendicular drawn from the centre (1, 2, 3) to the plane 2x-y+2z = 6

Length of the perpendicular

$$= \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{(2)(-1) + (-1)(2) + (2)(3) - 6}{\sqrt{4 + 1 + 4}}$$

$$= \frac{-2 - 2 + 6 - 6}{\sqrt{9}} = \frac{-4}{3} = r$$

We know that $r = \sqrt{u^2 + v^2 + w^2 - d}$

$$r^2 = u^2 + v^2 + w^2 - d$$

$$d = u^2 + v^2 + w^2 - r^2$$

$$= (-1)^2 + (2)^2 + (3)^2 - \left(\frac{-4}{3}\right)^2$$

$$= 1 + 4 + 9 - \frac{16}{9} = 14 - \frac{16}{9} = \frac{110}{9}$$

$$(2) \Rightarrow x^2 + y^2 + z^2 + 2x - 4y - 6z + \frac{110}{9} = 0$$

$$9\left(x^2 + y^2 + z^2\right) + 18x - 36y - 54z + 110 = 0$$

Problem 4: Find the equation of the sphere having the points (-4, 5, 1) and (4, 1, 7) as ends of a diameter.

Solution: Formula: $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

Therefore the equation of the required sphere is

$$(x + 4)(x - 4) + (y - 5)(y - 1) + (z - 1)(z - 7) = 0$$

$$x^2 + y^2 + z^2 - 6y - 8z - 4 = 0$$

Problem 5: Check whether the two spheres

$x^2 + y^2 + z^2 + 6y + 2z + 8 = 0$ *and* $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$ intersect each other orthogonally.

Solution: Given

$$x^2 + y^2 + z^2 + 6y + 2z + 8 = 0 \qquad (1)$$

$$x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0 \qquad (2)$$

Here $u_1 = 0, v_1 = 3, w_1 = 1, d_1 = 8$

$u_2 = 3, v_2 = 4, w_2 = 2, d_2 = 20$
Condition for orthogonal spheres is $2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$
L.H.S = 0 + 24 + 4 = 28
R.H.S = 8 + 20 = 28
L.H.S = R.H.S
Hence the two spheres intersect orthogonally.

Problem 6: Find the equation of the sphere with centre at (2, 3, 5), which touches the XOY plane.

Solution: Let $(x_1, y_1, z_1) = (2,3,5)$

Formula: Radius = perpendicular distance from (x_1, y_1, z_1) to the plane ax + by + cz +d = 0

$$= \pm \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$$

Radius = perpendicular distance from (2, 3, 5) to the plane z = 0

$$= \pm \frac{5}{\sqrt{0^2 + 0^2 + 1^2}} = \pm 5$$

The required sphere is $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$

$$(x-2)^2 + (y-3)^2 + (z-5)^2 = 5^2$$

$$x^2 - 4x + 4 + y^2 - 6y + 9 + z^2 - 10z + 25 = 25$$

$$x^2 + y^2 + z^2 - 4x - 6y - 10z + 13 = 0$$

Problem 7: Find the equation of the cone with vertex at the origin and passing through the curve $x^2 + y^2 = 9$, z = 3

Solution:
z = 3 implies z/3 = 1
Homogenizing $x^2 + y^2 = 9$, we get $x^2 + y^2 = 9.1^2 = 9.(z/3)^2$
i.e., $x^2 + y^2 = z^2$
This is the equation of the required cone.

Problem 8: Find the equation of the cone whose vertex is the origin and guiding curve is $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1, \quad x + y + z = 1$

Solution:

The required equation of the cone is obtained by homogenizing $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1$ with $x + y + z = 1$.

i.e., $$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1^2 = (x + y + z)^2$$

i.e., $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{1}=x^2+y^2+z^2+2xy+2yz+2zx$

i.e., $9x^2+4y^2+36z^2=36(x^2+y^2+z^2+2xy+2yz+2zx)$

i.e., $27x^2+32y^2+72(xy+yz+zx)=0$

Problem 9: Show that the line $\frac{x}{1}=\frac{y}{m}=\frac{z}{n}$ subject to $1^2+m^2-4n^2=0$ generates the cone $x^2+y^2-4z^2=0$

Solution:

The line $\frac{x}{1}=\frac{y}{m}=\frac{z}{n}$ passes through the origin. Hence origin is the vertex of the required cone. Also the direction ratios l, m and n should satisfy the equation of the cone. $1^2+m^2-4n^2=0$ implies l, m, n satisfy the equation $x^2+y^2-4z^2=0$. Hence the equation of the required cone is $x^2+y^2-4z^2=0$.

Problem 10: If $\frac{x}{1}=\frac{y}{1}=\frac{z}{k}$ is a generator of the cone $x^2+y^2-z^2=0$, find the value of k.

Solution:

Origin is the generator of the cone. The direction ratios 1, 1, k of the generator should satisfy the equation of the cone. Therefore, $1^2+1^2-k^2=0$. i.e., $k=\pm\sqrt{2}$

Problem 11: Find the equation of the right circular cone whose vertex is the origin, whose axis is the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and which has semi-vertical angle of 30°.

Solution: Let a generator of the cone be $\frac{x}{1}=\frac{y}{m}=\frac{z}{n}$, where l, m, n are its direction ratios.

Direction ratios of the axis are 1, 2, 3.

Therefore $\cos 30=\frac{l+2m+3n}{\sqrt{1^2+2^2+3^2}\sqrt{l^2+m^2+n^2}}$

$$\cos^2 30=\frac{(l+2m+3n)^2}{14(l^2+m^2+n^2)}$$

$$14(l^2+m^2+n^2)(3/4)=(l+2m+3n)^2$$

$$42(l^2+m^2+n^2)=4(l+2m+3n)^2$$

Hence the equation of the cone is

$$42(x^2+y^2+z^2)=4(x+2y+3z)^2$$

i.e., $19x^2+13y^2+3z^2-8xy-42yz-12zx=0$

Problem 12: Find the equation of the cone of the second degree which passes through the axes

Solution: The cone passes through the axes. Therefore the verrtex of the cone is the origin. The equation of the cone is homogeneous of second degree in x, y and z.

i.e., $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ (1)

Given that x-axis is a generator. Then y = 0, z = 0 must satisfy the equation (1). Therefore, a = 0. Similarly, y and z axes are generators imply that b = 0 and c = 0

Hence the equation of the cone is $fyz + gzx + hxy = 0$

Problem 13: Find the right circular cylinder, whose axis is z-axis and radius a.

Solution: Let $P(x_1, y_1, z_1)$ be any point on the surface of the cylinder. Draw PM perpendicular to the z-axis. Then PM = a and OM = z_1, where O is the origin.

$OP^2 = OM^2 + PM^2$

i.e., $x_1^2 + y_1^2 + z_1^2 = z_1^2 + a^2$

i.e., $x_1^2 + y_1^2 = a^2$

Locus of (x_1, y_1, z_1) is $x^2 + y^2 = a^2$, which is the equation of the required cylinder.

Problem 14: Write down the equation of the right circular cylinder whose axis is the straight line $\frac{x-\alpha}{1} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and whose radius is a

Solution: $$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = \left[\frac{(x-\alpha)l + (y-\beta)m + (z-\gamma)n}{\sqrt{l^2+m^2+n^2}}\right]^2 + a^2$$

is the required equation of the right circular cylinder.

Problem 15: What is the general equation of a cylinder whose generators are parallel to the z-axis?

Solution: The general equation of a cylinder whose generators are parallel to the z-axis is f(x, y) = $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.