XI) LAGRANGE'S THEOREM :

Let G be a finite group of order 'n' & H be any subgroup of G. Then the order of H divides the order of G, (ie) $O(H) / O(G)$

(or) The order of each subgroup of a finite group is a divisor of the order of the group.

Proof: Let $(G, *)$ be a group whose order is n.

(ie) $O(G) = n$.

Let $(H, *)$ be a subgroup of G whose order is m.

(ie) $O(H) = m$

Let $h_1, h_2, h_3, \ldots, h_m$ be the 'm' different elements of H.

The right coset $H*a$ of H in G is defined by

$H*a = \{h_1 * a, h_2 * a, \ldots, h_m * a\}, \ a \in G$

Since there is a one-one correspondence btwn. the elts. of H and $H*a$, the elts of $H*a$ are distinct.

Hence each right coset of H in G has 'm' distinct elts.

WKT any right cosets of H in G are either disjoint or identical

The no. of distinct right cosets of H in G are either is finite (say k) [$\because$ G is finite]

The union of these k distinct cosets of H in G is equal to G

Let these k distinct right cosets be

$$H * a_1, \; H * a_2, \; H * a_3, \; \ldots, \; H * a_k$$

then $G = (H * a_1) \cup (H * a_2) \cup \ldots \cup (H * a_k)$

$$\therefore \; O(G) = O(H * a_1) + O(H * a_2) + \ldots + O(H * a_k)$$

$$n = m + m + \ldots + m \quad (k \text{ times})$$

$$n = km$$

$$\Rightarrow k = \frac{n}{m} \quad \text{(ie)} \quad \frac{O(G)}{O(H)} = k$$

Since k is an integer (+ive), m is a divisor of n.

$\Rightarrow O(H)$ is a divisor of $O(G)$

$\Rightarrow O(H)$ divides $O(G)$.