SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution) COIMBATORE – 35

23MAT201 –PARTIAL DIFFERENTIAL EQUATIONS AND TRANSFORMS

UNIT-III : APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

PART-A

1. **What is the various solution of the one-dimensional wave equation? (or) What are the various solution of $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$?**

   Ans:

   i) $y(x,t) = (A_1 e^{px} + A_2 e^{-px})(A_3 e^{cpt} + A_4 e^{-cpt})$

   ii) $y(x,t) = (A_1 cospx + A_2 sinpx)(A_3 coscpt + A_4 sincpt)$

   iii) $y(x,t) = (A_1 x + A_2)(A_3 t + A_4)$

2. **In the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ what does c^2 stand for? (or) What is the constant a^2 in the wave equation $u_{tt} = a^2 u_{xx}$?**

   Ans:

$$a^2 = \frac{T}{M} = \frac{\text{Tension}}{\text{Mass per unit length of the string}}$$

3. **A string is stretched fastened to two points l apart. Motion is started by displacing the string into the form $y = y_0 sin\frac{\pi x}{l}$ from which it is released at time $t = 0$. Formulate this problrm as a boundary value problem.**

   Ans:

   The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.

   The boundary and initial conditions are

   i) $y(0,t) = 0$ for all $t \geq 0$

   ii) $y(l,t) = 0$ for all $t \geq 0$

   iii) $(\frac{\partial y}{\partial t})_{(x,0)} = 0$

   iv) $y(x,0) = f(x) = y_0 sin\frac{\pi x}{l}$

4. **What conditions are assumed in deriving the one-dimensional wave equation?**

   Ans:

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$. In deriving this equation, we make the following assumptions

i) The motion takes place entirely in one plane.

ii) We consider only transverse vibrations, the horizontal displacement of the particles of the string is negligible.

iii) The tension T is constant at all times and at all points of the deflected string.

iv) T is considered to be so large compared with the weight of the string and hence the force of gravity is negligible.

v) The effect of string is negligible.

5. Write down the boundary and initial conditions for solving the vibration of string equation, if the string is subjected to initial displacement f(x) and initial velocity g(x).

Ans:

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.

The boundary and initial conditions are

i) $y(0,t) = 0$ for all $t \geq 0$

ii) $y(l,t) = 0$ for all $t \geq 0$

iii) $(\frac{\partial y}{\partial t})_{(x,0)} = 0$

iv) $y(x,0) = f(x)$

6. Write down all variable separable solutions of the one-dimensional heat equation $u_t = \alpha^2 u_{xx}$

Ans:

i) $u(x,t) = (A_1 e^{\lambda x} + A_2 e^{-\lambda X}) A_3 e^{\alpha^2 \lambda^2 t}$

ii) $u(x,t) = (A_1 cos\lambda x + A_2 sin\lambda x) A_3 e^{-\alpha^2 \lambda^2 t}$

iii) $u(x,t) = (A_1 x + A_2) A_3$

7. State the suitable solution of the one-dimensional heat equation.

Ans:

$$u(x,t) = (A_1 cos\lambda x + A_2 sin\lambda x) A_3 e^{-\alpha^2 \lambda^2 t}$$

8. State the governing equation for one dimensional heat equation and necessary conditions to solve the problem.

Ans:

The one-dimensional heat equation is $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$.

The boundary and initial conditions are

i) $u(0,t) = k_1{}^oC$ for all $t \geq 0$

ii) $u(l,t) = k_2{}^oC$ for all $t \geq 0$

iii) $u(x,0) = f(x), 0 < x < l$

9. State any two laws of one-dimensional heat equation.

Ans:

i) Heat flows higher to lower temperature.

ii) The rate at which heat flows across any area is proportional to the area and to the temperature gradient normal to the curve. This constant proportionality is known as the thermal conductivity (k) of the material. It is known as Fourier law of heat conduction.

10. In steady state conditions, derive the solution of one-dimensional heat flow equation.

Ans:

The P.DE. of one-dimensional heat flow is $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} ------(1)$

In steady state condition, the temperature u depends only on x and not on time t.

Hence $\frac{\partial u}{\partial t} = 0$

Then (1) reduces to $\frac{d^2 u}{dx^2} = 0 \ ----- (2)$

The general solution is $u = ax + b$ where a, b are arbitrary.

11. Write down the governing equation of two-dimensional steady state heat conduction.

Ans:

The required equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

12. Write down the three possible solutions of Laplace equation in two-dimensions.

Ans:

Laplace equation in two-dimensions is $\nabla^2 u = 0$

i) $u(x,t) = (A_1 e^{px} + A_2 e^{-px})(A_3 cospy + A_4 sinpy)$

ii) $u(x,t) = (A_1 cospx + A_2 sinpx)(A_2 e^{px} + A_4 e^{-px})$

iii) $u(x,t) = (A_1 x + A_2)(A_3 y + A_4)$

13. The ends A and B of a rod of length 10cm have their temperature kept at $20^\circ C$ and $70^\circ C$. Find the steady state temperature distribution on the rod.

Ans:

The steady state temperature distribution on the rod

$$u(x) = \left(\frac{b-a}{l}\right)x + a \quad -------- (1)$$

Here, a: Temperature at the end $x = 0$, i.e., $a = 20^{\circ}C$

b: Temperature at the end $x = l$, i.e., $b = 70^{\circ}C$

l: Length of the rod, i.e., $l = 10cm$

(1)$\Rightarrow u(x) = \left(\frac{70-20}{10}\right)x + 20 \; = \; 5x + 20$

14. Classify the partial differential equation $u_{xx} + 2u_{xy} + u_{yy} = 0$

Ans:

Given: $u_{xx} + 2u_{xy} + u_{yy} = 0$

Second order p.d.e in the function u of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) = 0$$

Here $A = 1, B = 2, C = 1$

$B^2 - 4AC = (2^2) - 4(1)(1) \; = 0$

The given equation is parabolic equation.

15.Classify the partial differential equation $2x\frac{\partial^2 u}{\partial x^2} + 4x\frac{\partial^2 u}{\partial x \partial y} + 8x\frac{\partial^2 u}{\partial y^2} = 0$

Ans:

Given: $2x\frac{\partial^2 u}{\partial x^2} + 4x\frac{\partial^2 u}{\partial x \partial y} + 8x\frac{\partial^2 u}{\partial y^2} = 0$

Second order p.d.e in the function u of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) = 0$$

Here $A = 2x, B = 4x, C = 8x$

$B^2 - 4AC = (2^2x) - 4(2x)(8x) \; = -60x^2$

If $x = 0$, we get $B^2 - 4AC = 0$, the given equation is a parabolic equation.

If $x < 0 \; (or) \; x > 0$, we get $B^2 - 4AC = -ve < 0$, the given equation
is an elliptic equation.