UNIT 4 Fourier Transforms
Fourier Transform pair

Show that the fourier transform of

$$f(x)=\begin{cases} a^2-x^2, & |x|<a \\ 0, & |x|>a>0 \end{cases}$$ and hence find that

$$2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]$$. Hence deduce that

$$\int_0^\infty\left(\frac{\sin t - t\cos t}{t^3}\right)dt = \frac{\pi}{4}$$. using P.I show that

$$\int_0^\infty\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \pi/15$$

$$f(x)=\begin{cases} a^2-x^2, & -a<x<a \\ 0, & -\infty<x<-a \ \& \ a<x<\infty \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}(a^2-x^2)\cos sx\,dx + i\int_{-a}^{a}(a^2-x^2)\sin sx\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_0^a(a^2-x^2)\cos sx\,dx + i(0)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[(a^2-x^2)\frac{\sin sx}{s} - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-2a\frac{\cos sa}{s^2} + \frac{2\sin sa}{s^3}\right] = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - as\cos sa}{s^3}\right]$$

put $a=1$ $\quad F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$

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i) Using Inverse Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3}\right] \left[\cos sx - i\sin sx\right] ds$$

$$= \frac{2}{\pi} \cdot 2\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds$$

Put $x=0$ $\quad f(0) = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\, ds$

$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) ds = \frac{\pi}{4} f(0) = \frac{\pi}{4}(1-0) = \frac{\pi}{4}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}.$$

ii) Using Parseval's Identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-x^2)^2 dx = \int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right)\right]^2 ds$$

$$2\int_{0}^{1} (1+x^4-2x^2)\, dx = \frac{16}{\pi} \int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$\frac{16}{\pi} \int \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$= 2\left[1 + \frac{1}{5} - \frac{2}{3}\right]$$

$$= 2\left[\frac{15+3-10}{15}\right] = \frac{16}{15}$$

$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}\left(\frac{\pi}{16}\right)$$

$$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$$